On the Hermitian K-Theory of the Stable Envelope

by

Daniel Marlowe

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Contents

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Mathematics. It has been composed by myself and has not been submitted in any previous application for any degree. A (strict) subset of the results in this thesis appear in pre-print form [MS24], coauthored with my supervisor, Dr. Marco Schlichting; the work presented was carried out by the author.

Abstract

We exhibit a natural equivalence between the Grothendieck-Witt space associated to an exact form category, and that of its derived Poincaré category, and show that these equivalences deloop to an equivalence of (non-connective) Grothendieck-Witt spectra. In contrast to existing approaches, our argument proceeds via an analysis of an explicit model for the derived functor of a quadratic functor on a certain class of well-behaved (complicial) exact categories with weak equivalences (which includes categories of complexes and pretriangulated dg-categories), showing that hermitian constructions on these descend along the localisation functor in a way that preserves the Grothendieck-Witt theory. We also demonstrate that the genuine symmetric Poincaré structure on the bounded derived ∞-category of an ordinary exact category, defined with respect to the Postnikov t-structure on its ind-category following [CHN25], coincides with the derived Poincaré structure of on-the-nose symmetric forms; this gives in particular another proof of the comparison of *ibid.* of on-the-nose and genuine symmetric Grothendieck-Witt theory of quasi-separated divisorial schemes. We finish with a sketch of a possible approach to a theorem of the heart for the L-theory of stable ∞-categories equipped with bounded heart structures.

Notation

We list here notation which commonly appears in the body of the thesis.

Symbol	Meaning
$\mathcal{S}$	the ∞-category of (small) spaces (homotopy types)
$\mathcal{S}\mathrm{p}$	the ∞-category of (small) spectra
$\mathcal{C}\mathrm{at}_\infty$	the (large) ∞-category of small ∞-categories
$\mathcal{C}\mathrm{AT}_\infty$	the (very large) ∞-category of large ∞-categories
$\mathcal{P}\mathrm{r}^{\mathrm{L}}$	the (very large) ∞-category of presentable ∞-categories and left adjoints
$\mathcal{S}\mathrm{et}$	the ordinary category of (small) sets
$\mathrm{s}\mathcal{S}\mathrm{et}$	the ordinary category of (small) simplicial sets (with the Kan-Quillen model structure)
$\mathrm{s}\mathcal{C}$	the ∞-category of simplicial objects in $\mathcal{C}$
$\mathrm{s}\mathcal{S}$	the ∞-category of (small) simplicial spaces
$\mathcal{P}(\mathcal{C})$	the ∞-category of presheaves of spaces on $\mathcal{C}$
$\mathcal{P}_\Sigma(\mathcal{C})$	the ∞-category of product-preserving presheaves of spaces on $\mathcal{C}$
$\mathcal{P}_{\mathrm{lex}}(\mathcal{E})$	the ∞-category of left-exact presheaves of spaces on an exact ∞-category $\mathcal{E}$
$\mathcal{P}_\Sigma^{\mathcal{S}\mathrm{p}}(\mathcal{C})$	the ∞-category of product-preserving presheaves of spectra on $\mathcal{C}$
$\mathcal{P}_{\mathrm{lex}}^{\mathrm{st}}(\mathcal{C})$	the ∞-category of left-exact presheaves of spectra on an exact ∞-category $\mathcal{E}$
$\mathcal{S}\mathrm{h}_\Sigma(\mathcal{C})$	the ∞-category of product-preserving sheaves of spaces on an ∞-site $\mathcal{C}$
$\mathcal{S}\mathrm{h}_\Sigma^{\mathcal{S}\mathrm{p}}(\mathcal{C})$	the ∞-category of product-preserving sheaves of spectra on an ∞-site $\mathcal{C}$
$\int_{\mathcal{C}} \mathsf{F}$	the cartesian unstraightening of a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ or $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$
$\int^{\mathcal{C}} \mathsf{F}$	the cocartesian unstraightening of a functor $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ or $\mathcal{C} \to \mathcal{S}$
$\mathsf{F} \dashv \mathsf{G}$	an adjoint pair of functors, with F left adjoint to G
$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$	the ∞-category of functors $\mathcal{C} \to \mathcal{D}$
$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$	the ∞-category of exact functors of exact ∞-categories $\mathcal{C} \to \mathcal{D}$
$\mathcal{E}\mathrm{xact}_\infty$	the ∞-category of (small) exact ∞-categories and exact functors
$\mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$	the ∞-category of (small) stable ∞-categories and exact functors

$(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$	(co)connective aisles in a stable ∞-category equipped with a t-structure
$(\tau_{\leq 0}, \tau_{\geq 0})$	truncation functors for a stable ∞-category equipped with a t-structure
$\mathcal{C}^{t\heartsuit}$	the t-heart of a stable ∞-category equipped with a t-structure
$\mathcal{C}^{w\heartsuit}$	the weighted heart of a stable ∞-category equipped with a weight structure
$\mathcal{C}^{h\heartsuit}$	the heart of a stable ∞-category equipped with a heart structure
$\mathrm{Map}(-,-) = \mathrm{Map}_{\mathcal{C}}(-,-)$	the mapping space bifunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ for an ∞-category $\mathcal{C}$
$\mathrm{Hom}(-,-) = \mathrm{Hom}_{\mathcal{C}}(-,-)$	the mapping space bifunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ for an additive ∞-category $\mathcal{C}$
$\mathrm{hom}(-,-) = \mathrm{hom}_{\mathcal{C}}(-,-)$	the mapping spectrum bifunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}p$ for a stable ∞-category $\mathcal{C}$
$\mathcal{C}^{\simeq}$	the maximal sub-∞-groupoid of an ∞-category $\mathcal{C}$ (the groupoid core)
$\lvert\mathcal{C}\rvert$	the total localisation of an ∞-category $\mathcal{C}$
$\mathrm{Ho}(\mathcal{C})$	the homotopy category of $\mathcal{C}$
$(\mathcal{E}, w, \mathbb{D}, \eta, Q)$	exact form category with weak equivalences, with underlying exact category $\mathcal{E}$, and abelian-group valued quadratic functor $Q : \mathcal{E}^{op} \to \mathcal{A}b$
Q_{nd}	the subfunctor of nondegenerate forms of Q
$(\mathcal{C}, \Qoppa)$	hermitian or Poincaré category with underlying stable ∞-category $\mathcal{C}$, and quadratic functor $\Qoppa : \mathcal{C}^{op} \to \mathcal{S}p$
$(\mathcal{C}, \Qoppa^{[n]})$	the shifted hermitian structure $\Sigma^n \circ \Qoppa : \mathcal{C}^{op} \to \mathcal{S}p$
$\Qoppa_{nd}$	the subfunctor $\Qoppa_{nd} \subset \Omega^\infty \Qoppa$ of nondegenerate forms of $\Qoppa$
$\mathcal{C}_{/x} = (\mathcal{C} \downarrow x)$	the slice (alias comma) category of $\mathcal{C}$ over an object x
$\mathcal{C}_{x/} = (x \downarrow \mathcal{C})$	the slice (alias comma) category of $\mathcal{C}$ under an object x

1 Introduction

Hermitian K-theory is at heart the study of the stable algebra of projective modules equipped with a generalised notion of quadratic form. It is the algebraic analogue of Real topological K-theory, and a quadratic refinement of algebraic K-theory. The construction and properties of higher hermitian K-groups for rings were initially laid out by Karoubi [Kar73], culminating in particular in the proof of the celebrated *fundamental theorem* [Kar80] relating algebraic and hermitian K-theory. Through work of Giffen, Hornbostel [Hor05], and Schlichting [HS04], [Sch10a], [Sch10b], among others, these results were generalised to the setting of exact categories with duality, and later to dg-categories with duality and uniquely 2-divisible mapping complexes [Sch17]. As the latter restriction suggests, the prime 2 is of particular importance in governing the behaviour of hermitian K-theory: given a ring R in which 2 is a unit, the map associating to a quadratic form on a finitely generated projective R-module P its bilinear polarisation gives an isomorphism between the sets of quadratic and symmetric bilinear forms on P, but this fails in general, as does the invariance of hermitian K-theory under a naïve version of derived equivalence (see [Sch17, Prop. 2.1]). The foundations of hermitian K-theory were revisited accordingly for more general notions of form and with no assumption on the invertibility of 2 in [Sch21], [Sch24b], and more recently, in pioneering work [CDH⁺I], [CDH⁺II], [CDH⁺III] of Calmès-Dotto-Harpaz-Hebestreit-Land-Moi-Nardin-Nikolaus-Steimle building on Lurie's formalism of Poincaré ∞-categories [Lur11].

The Poincaré formalism has proved fruitful in categorifying the relationship between hermitian K-theory and L-theory and algebraic surgery, and exhibits the fundamental fibre sequence relating these as a consequence of the isotropy separation square for the Real algebraic K-theory C_2-spectrum KR [CDH⁺II, §4.5]. Importantly, in this setting the hermitian K-functor GW enjoys a universal property as the initial grouplike additive spectrum-valued functor on the ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{p}}_{\infty}$ of Poincaré categories, whose underlying infinite loopspace receives a natural transformation from $\mathrm{Pn} : \mathcal{C}\mathrm{at}^{\mathrm{p}}_{\infty} \to \mathcal{S}$, where here $\mathrm{Pn}(\mathcal{C}, \Omega)$ is in an appropriate sense the moduli space of nondegenerate forms on a Poincaré category $(\mathcal{C}, \Omega)$, playing a role analogous to that of the groupoid core for algebraic K-theory.

As an invariant of algebraic varieties, hermitian K-theory has come to occupy a celebrated place in motivic homotopy theory [MV99]: fundamental work of Morel [Mor12] (see also [RSØ19]) identifies $\mathrm{GW}_0(k)$ with the endomorphisms ring $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$ of the motivic sphere spectrum over a perfect field k of characteristic not equal to 2. As such, $\mathrm{GW}_0(k)$ plays a role in motivic homotopy theory over Spec(k) analogous to that of the integers in usual homotopy theory, and is the universal receptacle for $\mathbb{A}^1$-Euler characteristics and Brouwer degrees. Computations of hermitian K-groups have been fundamental in the Asok-Fasel vector bundle classification program [AF14], [AF23], permitting arguments by obstruction theory in motivic homotopy, and more recently in work of Schäppi [Sch24a] providing a counterexample to the Hermite ring conjecture. As early as the 1970s, hermitian K-theory computations were used by Karoubi [Kar74] to give a lower bound on the order of $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$, refining a conjecture of Lichtenbaum. By work of Hornbostel when 2 is invertible, and

Calmès-Harpaz-Nardin [CHN25] in the general setting, (homotopy) symmetric hermitian K-theory (over some regular noetherian base scheme S) satisfies Nisnevich descent, $\mathbb{A}^1$-invariance, and a projective bundle formula, and is accordingly represented by a motivic spectrum KQ_S; moreover, recent work of Hoyois-Land [LH25] demonstrate motivic representability of genuine and classical variants of hermitian K-theory in the category of motivic spectra of Annala-Hoyois-Iwasa.

An ahistorical account of the development of hermitian K-theory might go as follows: one studies the zeroth Grothendieck-Witt group $GW_0^q(k)$ of quadratic forms attached to a field k, obtained as the group-completion of a certain abelian monoid of isometry classes of nondegenerate forms under orthogonal sum, and via a plus-construction style definition extends this to higher groups amenable to study by K-theoretic techniques. One then notes that this space is an invariant of additive (or exact, or dg-) categories with duality, and via a hermitian analogue of the Waldhausen $S_\bullet$-construction defines a Grothendieck-Witt space that extends the 'direct sum' group completion above.

A point of departure from vanilla K-theory is that one may consider different notions of form ((anti-)symmetric, symplectic, (anti-)quadratic...), in general giving rise to inequivalent Grothendieck-Witt spaces. These are in some sense parametrised by the norm map sending a quadratic form to its bilinear polarisation. One is then led to consider the form as part of the structure, leading to a Grothendieck-Witt theory of categories equipped with a nondegenerate quadratic functor encoding some notion of form. In the $\mathbb{Z}$-linear setting, these categories are exact, and the form functors are valued in abelian-groups; in the modern ($\mathbb{S}$-linear) setting, these categories are stable ∞-categories, and the form functors are reduced, 2-excisive, and take values in spectra. In the derived approach, hermitian K-theory enjoys a universal property most naturally phrased in the higher categorical language.

Successive generalisations of the K-theoretic apparatus have been accompanied by comparison results, namely Quillen's '$+ = \mathcal{Q}$' theorem [Gra76], the Gillet-Waldhausen theorem [TT90, Th. 1.11.7], and in the higher categorical setting [Bar13, Cor. 10.10, 10.16] and [BGT13, Cor. 7.12]. This thesis addresses the lack of a comparison in full generality between the Quillen-Waldhausen-style hermitian K-theory of exact categories [Sch21] and its higher categorical analogue [CDH⁺I], and provides the desired invariance under derived equivalence. The main result is the following (Theorem 5.3.6 in the main text).

Theorem A. *For* $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ *a complicial exact form category with weak equivalences, the Dwyer-Kan localisation* $\mathcal{E} \to L_w(\mathcal{E})$ *is a stable ∞-category, and carries a canonical Poincaré structure* $(L_w(\mathcal{E}), \mathbf{R}Q)$ *such that the localisation functor induces a natural equivalence of (nonconnective) hermitian* K*-theory spectra*

$$GW(\mathcal{E}, Q, w) \xrightarrow{\simeq} GW(L_w(\mathcal{E}), \mathbf{R}Q),$$

where the left- and right-hand sides denote respectively the form-categorical and Poincaré-categorical hermitian K*-theory spectra.*

In conjunction with the main result of [Sch24b], we obtain the following, generalising [CDH⁺II, Cor. B.2.6] and partly generalising the main result of [HS25] (the latter work concerns additive (alias split-exact) ∞-categories, while our results concern not-necessarily-split ∞-categories with discrete mapping spaces.

Theorem B. *For* $(\mathcal{E}, \mathbb{D}, \eta, Q)$ *an exact form category, the canonical embedding* $\mathcal{E} \hookrightarrow D_b(\mathcal{E})$ *into the bounded derived ∞-category induces an equivalence of spectra*

$$GW(\mathcal{E}, Q) \xrightarrow{\simeq} GW(D_b(\mathcal{E}), \Omega),$$

for $(\mathrm{D_b}(\mathcal{E}), \Omega)$ *the derived Poincaré category associated to* $(\mathcal{E}, \mathrm{Q})$.

As a corollary of this, we exhibit a canonical equivalence between the 1-categorical symmetric Grothendieck-Witt theory of an exact category with duality $\mathcal{E}$ and the genuine symmetric Grothendieck-Witt theory of the derived ∞-category $\mathrm{D_b}(\mathcal{E})$. This may be known to experts, but to our knowledge has been recorded only in the split-exact case.

Corollary. *Let* $(\mathcal{E}, \mathbb{D}, \eta)$ *be an exact category with strong duality. Then the derived Poincaré category of the symmetric form category* $(\mathcal{E}, \mathrm{Q}^s)$ *canonically identifies with the genuine symmetric structure* $(\mathrm{D_b}(\mathcal{E}), \Omega^{gs})$*, and the embedding* $\mathcal{E} \hookrightarrow \mathrm{D_b}(\mathcal{E})$ *induces an equivalence of hermitian* K*-theory spectra*

$$\mathrm{GW}(\mathcal{E}, \mathrm{Q}^s) \to \mathrm{GW}(\mathrm{D_b}(\mathcal{E}), \Omega^{gs}).$$

1.1 Relation to other work

Partial comparison results exist in the literature in the setting of additive form categories [HS25], and in the symmetric forms case for 2 invertible [CDH+II, App. B]. The former comparison follows from a detailed study of algebraic surgery on cobordism categories, incorporating in an essential manner the notion of a weight structure on a stable ∞-category. The corresponding theorem of the weighted heart [HS25, Cor. A.2.8] states that for a Poincaré category $(\mathcal{C}, \Omega)$ equipped with an exhaustive weight structure of dimension 0, the inclusion of the heart (which can be considered an additive form ∞-category) induces an identification

$$\mathrm{Pn}(\mathcal{C}^{w\heartsuit}, \Omega|_{\mathcal{C}^{w\heartsuit}})^{\mathrm{grp}} \xrightarrow{\simeq} \mathcal{GW}(\mathcal{C}, \Omega) \tag{1.1}$$

as soon as $\Omega|_{\mathcal{C}^{w\heartsuit}}$ takes values in connective spectra. When $\mathcal{C}^{w\heartsuit}$ has discrete mapping spaces, and $\Omega|_{\mathcal{C}^{w\heartsuit}}$ takes values in discrete spectra, the pair $(\mathcal{C}^{w\heartsuit}, \Omega|_{\mathcal{C}^{w\heartsuit}})$ is an additive form category, and by the main result of [Sch21] the group completion on the left coincides with the 'classical' Grothendieck-Witt space $\mathcal{GW}(\mathcal{C}^{w\heartsuit}, \Omega_{\mathcal{C}^{w\heartsuit}})$.
That the surgery techniques of [HS25] are limited to split-exact categories follows from the observation that an exact sequence in the heart of a weight structure necessarily splits. It is worth mentioning that the additive comparison (1.1) suffices to compare, for instance, the symmetric Grothendieck-Witt spaces associated to affine schemes, and via Zariski descent to divisorial schemes; see [CHN25, Prop. 4.6.1].
Our argument proceeds instead along the lines of [CDH+II, App. B], starting with a complicial exact form category with weak equivalences $(\mathcal{E}, \mathrm{Q}, w, \mathbb{D}, \eta)$ in the sense of [Sch21] and deriving the relevant structures to obtain a Poincaré category $(\mathrm{L}_w(\mathcal{E}), \mathbf{R}\mathrm{Q})$ in the sense of [CDH+I]. In doing so, we obtain an explicit description of the underlying infinite loopspace of the nonabelian derived functor associated with the quadratic functor Q, upon which the comparison theorem hinges. The stipulation that $\mathcal{E}$ be complicial is essentially a homotopical soundness condition, ensuring that $\mathcal{E}$ is an ∞-category of fibrant objects in the sense of [Cis19]; any exact category admits a fully faithful exact functor into a complicial exact category with weak equivalences under which the hermitian K-theory space is invariant, cf. §2.2.

1.2 Organisation

We give a brief summary of the contents of the thesis: §3 is concerned with the study of nonabelian derived functors on complicial exact categories, serving as a technical foundation for the comparison of Grothendieck-Witt spaces in §4. In §5, we upgrade the comparison to (non-connective) Grothendieck-Witt spectra in the sense of [Sch24b, §11] and [CDH⁺II, §4.2]. In §6, we give a brief treatment of genuine Poincaré structures, showing that the definition in [CHN25] recovers precisely the derived Poincaré structures of exact form categories. For the reader unfamiliar with complicial exact categories, we include a detailed survey in Appendix A.3, alongside technical results on exact categories in A.1 and A.2. Appendix A.4 is a brief recollection on underlying ∞-categories of model categories, relevant only to the comparison of homotopy (co)limits with ∞-(co)limits. Appendix B includes all of the results we use on n-polynomial and n-excisive functors, and Appendix C is a sketch of a possible approach to a theorem of the heart (in the sense of [Sau23]) for the L-theory of exact ∞-categories, following the approach of Harpaz in the appendix of [HS25]. The results in this appendix are mostly straightforward generalisations of those of Harpaz, and even though the argument does not go through completely, we include it in the hope that it may appear in completed form in future work, or at the very least indicate what is currently missing in our (or at least my) understanding of algebraic surgery on heart categories.

1.3 Conventions

This paper presents a recipe for the extraction of homotopy-coherent data from algebraic structures. To give a uniform treatment of the various homotopical ideas, we make use of the quasi-categorical model of $(\infty, 1)$-categories as developed by Joyal [Joy08] and Lurie [HTT], [HA]. We refer to quasi-categories as ∞-categories, drawing a distinction between these and ordinary categories when necessary (implicitly identifying an ordinary category with its nerve). We refer to higher-categorical (co)limits as (co)limits, using the term homotopy (co)limit when we make explicit reference to model-categorical models for these. *Essentially unique* should be read as *unique up to a contractible space of choices*. For ∞-categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and functors $i : \mathcal{C} \to \mathcal{D}$, $f : \mathcal{C} \to \mathcal{E}$, we denote by $i_! f$ the left Kan extension of f along i, when this exists. For a functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories, we write the ∞-categorical slices $\mathcal{D}_{/f}$ and $\mathcal{D}_{f/}$ as $(\mathcal{D} \downarrow f)$ and $(f \downarrow \mathcal{D})$, noting that when $\mathcal{C}, \mathcal{D}$ are ordinary categories, these coincide with the nerve of the 1-categorical slice categories by [HTT, Rem. 1.2.9.6]. Finally, we grade our chain complexes homologically.

2 Hermitian preliminaries

In this chapter we give a fairly concise overview of the different flavours of Grothendieck-Witt theory we will consider in the sequel. The classical approach to the higher K-theory of forms has its naissance in seminal work of Karoubi, and is essentially concerned with a hermitian refinement of algebraic K-theory à la Quillen, in which one considers an exact category $\mathcal{E}$ equipped with a duality functor $\mathbb{D} : \mathcal{E}^{op} \to \mathcal{E}$, furnishing some notion of symmetric form on objects of $\mathcal{E}$, with the key assumption that $\mathcal{E}$ is linear over $\mathbb{Z}[\frac{1}{2}]$. Such a situation arises naturally, for instance, in the study of categories of projective modules over a commutative ring (equipped with the trivial involution). The Grothendieck-Witt space is defined via a hermitian analogue of the $\mathcal{Q}$-construction which takes the duality into account. Various π_0-phenomena[1] are shown to exist on the space level, and the theory enjoys deep links with algebraic K-theory, and the theory of algebraic surgery of Ranicki.

Various of celebrated results in Grothendieck-Witt theory fail to hold as stated naively over $\mathbb{Z}$, ultimately owing to the conflation of different types of form in the presence of $\frac{1}{2}$. The theory was accordingly revisited in the ($\mathbb{Z}$-linear) formalism of [Sch21], in which the notion of form under consideration is parametrised explicitly by a quadratic abelian-group valued functor (which determines the underlying duality), and which is set up over $\mathbb{Z}$. Most recently, hermitian K-theory has enjoyed a homotopy coherent incarnation in [CDH+I]-[CDH+IV], the framework of which comprises the data of a stable ∞-category equipped with a nondegenerate spectrum-valued quadratic functor, all of which takes place over the sphere spectrum. The ensuing renaissance of hermitian K-theory has led to a modern theory that subsumes and vastly generalises[2] previous approaches, for instance in a construction of real algebraic K-theory for Poincaré categories, with links to real trace invariants and L-theory; and motivic representability results over $\mathbb{Z}$, both in the $\mathbb{A}^1$-invariant setting [CHN25], and for derived schemes [LH25] in the realm of non-$\mathbb{A}^1$-invariant motivic spectra as introduced by Annala-Hoyois-Iwasa [AI25], [AHI24].

We begin by making some recollections on the formalism of [Sch21], [Sch24b], and [CDH+I], [CDH+II].

2.1 Recollections on form categories

To avoid bloating this section, we have relegated a thorough discussion of exact categories to Appendix A.1, on the contents of which this section depends.

Definition 2.1.1. An **exact category with weak equivalences and duality** $(\mathcal{E}, \mathbb{D}, \eta, w)$ is the data of a Quillen exact category [Qui73], equipped with the following data:

[1] E.g., the existence of forgetful and hyperbolic maps rendering the pair $(K_0(\mathcal{E}), GW_0(\mathcal{E}, \vee))$ a C_2-Mackey functor, the exact sequence $K_0(\mathcal{E})_{C_2} \to GW_0(\mathcal{E}, \vee) \to L_0(\mathcal{E}, \vee) \to 0$...

[2] Demonstrating that this is so is the point of this thesis.

(i) A wide subcategory $w\mathcal{E} \subset \mathcal{E}$ of weak equivalences, satisfying 2-of-3, and closed under isomorphism and retracts in $\mathrm{Ar}(\mathcal{E})$, and under pushouts along ingressions and pullbacks along egressions.

(ii) An exact functor $\mathbb{D} : \mathcal{E}^{op} \to \mathcal{E}$, the **duality**, which preserves weak equivalences, and a natural transformation $\eta : 1_{\mathcal{E}} \to \mathbb{D}\,\mathbb{D}^{op}$, the double dual identification, such that for each $x \in \mathcal{E}$, $\mathbb{D}(\eta_x) \circ \eta_{\mathbb{D}(x)} = 1_{\mathbb{D}(x)}$ (exhibiting $\mathbb{D}^{op}$ as left adjoint to $\mathbb{D}$).

The duality is said to be strict if η is the identity transformation. If w is the class of isomorphisms, we simply say $(\mathcal{E}, \mathbb{D}, \eta)$ is an exact category with duality. Moreover, in the case $\mathcal{E}$ is an additive category equipped with the split-exact structure, we say that $(\mathcal{E}, \mathbb{D}, \eta)$ is an additive category with duality.
A **form functor** $(F, \varphi) : (\mathcal{E}, \mathbb{D}, \eta, w) \to (\mathcal{E}', \mathbb{D}', \eta', w')$ between exact categories with weak equivalences and duality is the data of an exact functor $F : \mathcal{E} \to \mathcal{E}'$ (preserving congressions and weak equivalences), and a duality compatibility transformation $\varphi : F \circ \mathbb{D} \Rightarrow \mathbb{D}' \circ F^{op}$ such that the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta'_{F(x)}} & (\mathbb{D}')^2 F(x) \\ \downarrow{\scriptstyle F(\eta_x)} & & \downarrow{\scriptstyle \mathbb{D}'(\varphi_x)} \\ F(\mathbb{D}^2(x)) & \xrightarrow{\varphi_{\mathbb{D}(x)}} & \mathbb{D}' F(\mathbb{D}(x)) \end{array}$$

commutes for each $x \in \mathcal{E}$. A form functor (F, φ) is said to be nonsingular if φ is a natural weak equivalence.

Warning 2.1.2. We depart somewhat from the terminology of [Sch21] and [Sch24b] in referring to an exact form category with weak equivalences and strong duality in the sense of *ibid.* as simply an exact form category with weak equivalences (since we won't encounter the version in which η is not a natural equivalence). Similarly, by an exact category with weak equivalences and duality we mean a tuple $(\mathcal{E}, w, \mathbb{D}, \eta)$ such that η is a natural weak equivalence (or an isomorphism in the case of an exact category with duality).

Remark 2.1.3. Call an object $x \in \mathcal{E}$ acyclic if the map $0 \to x$ is in $w\mathcal{E}$. Then for $(\mathcal{E}, w)$ an exact category with weak equivalence, by [Sch24b, Lem. 7.1], an ingression $x \rightarrowtail y$ in $\mathcal{E}$ is a weak equivalence if and only if its cokernel is acyclic, and dually an egression $x \twoheadrightarrow y$ is a weak equivalence if and only if its kernel is acyclic. Accordingly, we see that $\mathcal{E}^w \subset \mathcal{E}$ is an exact subcategory closed under retracts in $\mathcal{E}$.

For $(\mathcal{E}, \mathbb{D}, \eta)$ an exact category with duality and objects $x, y \in \mathcal{E}$, there is a natural isomorphism

$$\sigma : \mathrm{Hom}_{\mathcal{E}}(-, \mathbb{D}(-)) \cong \mathrm{Hom}_{\mathcal{E}}(-, \mathbb{D}(-)) \circ t$$

of functors $\mathcal{E}^{op} \times \mathcal{E}^{op} \to \mathcal{A}b$, for $t : \mathcal{E}^{op} \times \mathcal{E}^{op} \to \mathcal{E}^{op} \times \mathcal{E}^{op}$ the flip autoequivalence $(x, y) \mapsto (y, x)$. Pointwise, this is

$$\sigma_{x,y} : \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y)) \to \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(x)), \quad f \mapsto \mathbb{D}(f) \circ \eta_y,$$

satisfying $\sigma_{y,x}\sigma_{x,y}(f) = f$. In particular, the functor $\mathcal{E}^{op} \to \mathcal{A}b$, $x \mapsto \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))$ refines to an object of $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{A}b)^{BC_2}$, and $\varphi \in \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2}$ is said to be a *symmetric form* on x. The assignment $x \mapsto \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2}$ assembles into a **quadratic left-exact functor**, the symmetric forms functor, which equips any exact category with duality with the canonical structure of an **exact form category**. The following is [Sch24b, Def. 7.2].

Definition 2.1.4. An **exact form category with weak equivalences** is the data of an exact category with weak equivalences[3] and duality $(\mathcal{E}, \mathbb{D}, \eta, w)$ equipped with a functor $Q : \mathcal{E}^{op} \to \mathcal{A}b$ and transfer and restriction natural transformations τ, ρ of functors $\mathcal{E}^{op} \to \mathcal{A}b$,

$$\mathrm{Hom}_{\mathcal{E}}(-, \mathbb{D}(-)) \xrightarrow{\tau} Q(-) \xrightarrow{\rho} \mathrm{Hom}_{\mathcal{E}}(-, \mathbb{D}(-)),$$

satisfying the following conditions:

(i) Q is additively quadratic: for each $x \in \mathcal{E}$, the diagrams

$$\mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)) \xrightarrow{\tau_x} Q(x) \xrightarrow{\rho_x} \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))$$

are C_2-equivariant with respect to the trivial C_2-action on $Q(x)$, and satisfy $\rho_x\tau_x = 1_x + \sigma_{x,x}$, and for each $f, g \in \mathrm{Hom}_{\mathcal{E}}(x, y)$ and $\xi \in Q(y)$,

$$(f + g)^\bullet(\xi) = f^\bullet(\xi) + g^\bullet(\xi) + \tau_x(\mathbb{D}(g)\rho_y(\xi)f),$$

where we write $f^\bullet := Q(f)$.

(ii) Q is additionally quadratic left exact: for each congression $x \overset{i}{\rightarrowtail} y \overset{p}{\twoheadrightarrow} z$ in $\mathcal{E}$, the map $p^\bullet : Q(z) \to Q(y)$ exhibits $Q(z)$ as the total kernel in abelian groups of the square

$$\begin{array}{ccc} Q(y) & \xrightarrow{\quad i^\bullet \quad} & Q(x) \\ \big\downarrow{\scriptstyle \mathbb{D}(i)_*\rho_y} & & \big\downarrow{\scriptstyle \rho_x} \\ \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(x)) & \xrightarrow{\quad i^* \quad} & \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)). \end{array} \tag{2.1}$$

Remark 2.1.5. For $\mathcal{E}$ an exact category and $Q : \mathcal{E}^{op} \to \mathcal{A}b$ a reduced functor, the **polarisation** B_Q is defined as the kernel

$$B_Q(x, y) := \ker\left(Q(x \oplus y) \to Q(x) \oplus Q(y)\right).$$

We follow the classical terminology of Eilenberg-Mac Lane [EL53] (see also [BGMN22, Def. 2.4]) in calling Q **additively quadratic** if its polarisation preserves direct sums in either variable. A pair $(\mathcal{E}, Q : \mathcal{E}^{op} \to \mathcal{A}b)$ with Q additively quadratic then defines an exact form category if there is a natural isomorphism $\theta_{x,y} : B_Q(x, y) \cong \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y))$ of functors $\mathcal{E}^{op} \times \mathcal{E}^{op} \to \mathcal{A}b$, for $\mathbb{D} : \mathcal{E}^{op} \to \mathcal{E}$ a duality determined up to isomorphism by B_Q via Yoneda, and additionally Q is quadratic left exact. The maps τ, ρ are determined in this setup by θ, as the composites

$$\mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)) \xrightarrow{\theta^{-1}_{x,x}} B_Q(x, x) \hookrightarrow Q(x \oplus x) \xrightarrow{\Delta^\bullet_x} Q(x)$$

and

$$Q(x) \xrightarrow{\nabla^\bullet_x} Q(x \oplus x) \twoheadrightarrow B_Q(x, x) \xrightarrow{\theta_{x,x}} \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)).$$

Remark 2.1.6. If $(\mathcal{E}, Q, \mathbb{D}, \eta)$ is an additive form category and $x \overset{i}{\rightarrowtail} y \overset{p}{\twoheadrightarrow} z$ is a split exact sequence in $\mathcal{E}$, then under the first hypothesis of Definition 2.1.4, the sequence

$$0 \to Q(z) \xrightarrow{p^\bullet} Q(y) \xrightarrow{(i^\bullet, -\mathbb{D}(i)_*\rho_y)} Q(x) \oplus \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(x)) \xrightarrow{(\rho_x, i^*)} \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)) \to 0$$

is exact, i.e. $Q(z) \to Q(y)$ exhibits $Q(z)$ as the total kernel in abelian groups (in fact in $\mathcal{S}p$) of the diagram (2.1); see [Sch21, Lem. A.12].

[3] If w is the class of isomorphisms, we simply refer to the tuple $(\mathcal{E}, Q, \mathbb{D}, \eta)$ as an exact form category, and if moreover $\mathcal{E}$ is an additive category equipped with the split-exact structure, we refer to $(\mathcal{E}, Q, \mathbb{D}, \eta)$ as an additive form category.

Example 2.1.7. As remarked above, for $(\mathcal{E}, \mathbb{D}, \eta)$ an exact category with duality, the functor

$$Q^s : \mathcal{E}^{op} \to \mathcal{A}b, \quad x \mapsto \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2}$$

equips $\mathcal{E}$ with the structure of an exact form category (see [Sch21, Ex. 2.24]): the polarisation B_{Q^s} is given by

$$\begin{aligned} B_{Q^s}(x, y) &= \ker\left[\mathrm{Hom}_{\mathcal{E}}(x \oplus y, \mathbb{D}(x \oplus y))^{C_2} \to \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2} \oplus \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(y))^{C_2}\right] \\ &\cong \ker\left[\mathrm{Hom}_{\mathcal{E}}(x \oplus y, \mathbb{D}(x \oplus y)) \to \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)) \oplus \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(y))\right]^{C_2} \\ &\cong \left[\mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y)) \oplus \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(x))\right]^{C_2} \cong \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y)). \end{aligned}$$

There is an analogous construction of the so-called quadratic forms functor Q^q on $\mathcal{E}$, given by taking C_2-orbits (and using that $B_Q(x, y)$ is equivalently the cokernel of the split inclusion $Q(x) \oplus Q(y) \to Q(x \oplus y)$). The quadratic forms functor is additively quadratic, but in general fails to be quadratic left exact if the exact structure on $\mathcal{E}$ is not split.

A form functor between exact form categories with weak equivalences and duality

$$(F, \varphi_q, \varphi) : (\mathcal{A}, Q_{\mathcal{A}}, \mathbb{D}_{\mathcal{A}}, \eta_{\mathcal{A}}, w_{\mathcal{A}}) \to (\mathcal{B}, Q_{\mathcal{B}}, \mathbb{D}_{\mathcal{B}}, \eta_{\mathcal{B}}, w_{\mathcal{B}})$$

is the data of a form functor $(F, \varphi) : (\mathcal{A}, \mathbb{D}_{\mathcal{A}}, \eta_{\mathcal{A}}, w_{\mathcal{A}}) \to (\mathcal{B}, \mathbb{D}_{\mathcal{B}}, \eta_{\mathcal{B}}, w_{\mathcal{B}})$ of exact categories with weak equivalences and duality in the sense of Definition 2.1.1, and $\varphi_q : Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ F^{op}$ a natural transformation such that for each $x \in \mathcal{A}$, the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{A}}(x, \mathbb{D}_{\mathcal{A}}(x)) & \xrightarrow{\tau_x} & Q_{\mathcal{A}}(x) & \xrightarrow{\rho_x} & \mathrm{Hom}_{\mathcal{A}}(x, \mathbb{D}_{\mathcal{A}}(x)) \\ \downarrow{\scriptstyle f \mapsto \varphi_x F(f)} & & \downarrow{\scriptstyle \varphi_{q,x}} & & \downarrow{\scriptstyle f \mapsto \varphi_x F(f)} \\ \mathrm{Hom}_{\mathcal{B}}(F(x), \mathbb{D}_{\mathcal{B}}(F(x))) & \xrightarrow{\tau_{F(x)}} & Q_{\mathcal{B}}(F(x)) & \xrightarrow{\rho_{F(x)}} & \mathrm{Hom}_{\mathcal{B}}(F(x), \mathbb{D}_{\mathcal{B}}(F(x))) \end{array} \tag{2.2}$$

commutes. If both $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ encode symmetric forms, a form functor reduces to the notion of a form functor between exact categories with weak equivalences and duality in the sense of Definition 2.1.1.
Write $\mathrm{w}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}$ for the category spanned by the exact form categories with weak equivalences, and nonsingular form functors between them, and $\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}$ for the full subcategory of exact form categories.

Remark 2.1.8. The data of a nonsingular form functor $(F, \varphi, \varphi_q) : (\mathcal{E}, Q, \mathbb{D}, \eta) \to (\mathcal{E}', Q', \mathbb{D}', \eta')$ between form categories is overspecified in the sense that the duality compatibility is determined by the pair (F, φ_q): there is a map

$$\psi_{x,y} : \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y)) \to \mathrm{Hom}_{\mathcal{E}'}(F(x), \mathbb{D}' F(y))$$

induced by the naturality of φ_q and the identification $\ker(Q(x \oplus y) \to Q(x) \oplus Q(y)) \cong \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(y))$. Setting $\varphi_x := \psi_{\mathbb{D}(x),x}\left(1_{\mathbb{D}(x)}\right)$ for each $x \in \mathcal{E}$, we see that for each map $f : x \to y$, the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{E}}(\mathbb{D}(y), \mathbb{D}(y)) & \xrightarrow{\mathbb{D}(f)_*} & \mathrm{Hom}_{\mathcal{E}}(\mathbb{D}(y), \mathbb{D}(x)) & \xleftarrow{\mathbb{D}(f)^*} & \mathrm{Hom}_{\mathcal{E}}(\mathbb{D}(x), \mathbb{D}(x)) \\ \downarrow{\scriptstyle \psi_{\mathbb{D}(y),y}} & & \downarrow{\scriptstyle \psi_{\mathbb{D}(y),x}} & & \downarrow{\scriptstyle \psi_{\mathbb{D}(x),x}} \\ \mathrm{Hom}_{\mathcal{E}'}(F\mathbb{D}(y), \mathbb{D}' F(y)) & \xrightarrow{\mathbb{D}' F(f)_*} & \mathrm{Hom}_{\mathcal{E}'}(F\mathbb{D}(y), \mathbb{D}' F(x)) & \xleftarrow{F\mathbb{D}(f)^*} & \mathrm{Hom}_{\mathcal{E}'}(F\mathbb{D}(x), \mathbb{D}' F(x)) \end{array}$$

witnesses $\mathbb{D}'F(f)\circ\varphi_y = \psi_{\mathbb{D}(y),x}(\mathbb{D}(f)) = \varphi_x\circ F\mathbb{D}(f)$, i.e. φ furnishes a natural transformation $F\mathbb{D} \Rightarrow \mathbb{D}'F^{op}$. Commutativity of the diagram (2.2) follows from that of

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}}(x,\mathbb{D}(x)) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{E}'}(F(x),F\mathbb{D}(x)) \\ & \searrow^{\psi_{x,x}} & \downarrow{\scriptstyle \varphi_{x,*}} \\ & & \mathrm{Hom}_{\mathcal{E}'}(F(x),\mathbb{D}'F(x)), \end{array}$$

which in turn follows from the Yoneda lemma (the natural transformations $\varphi_{x,*}\circ F$ and $\psi_{-,x}$, $\mathrm{Hom}_{\mathcal{E}}(-,\mathbb{D}(x)) \Rightarrow \mathrm{Hom}_{\mathcal{E}'}(F(-),\mathbb{D}'F(x)))$ each send $1_{\mathbb{D}(x)}$ to φ_x). The condition that $\mathbb{D}'(\varphi_x)\circ\eta_{F(x)} = \varphi_{\mathbb{D}(x)}\circ F(\eta_x)$ of [Sch21, §2] is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}}(\mathbb{D}(x),\mathbb{D}(y)) & \xrightarrow{\psi_{\mathbb{D}(x),y}} & \mathrm{Hom}_{\mathcal{E}'}(F\mathbb{D}(x),\mathbb{D}'F(y)) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathrm{Hom}_{\mathcal{E}}(y,\mathbb{D}^2(x)) & \xrightarrow{\psi_{y,\mathbb{D}(x)}} & \mathrm{Hom}_{\mathcal{E}'}(F(y),\mathbb{D}'F\mathbb{D}(x)), \end{array}$$

where the vertical isomorphisms arise from self-adjointness of $\mathbb{D}$.

Remark 2.1.9. It will be useful to have a construction of $\mathrm{w}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}$ closer to those of [CDH+I]. Write $\mathrm{w}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$ for the ordinary category of small exact categories with weak equivalences and exact functors between them, and CAT_1 for the (large) category of small categories. Consider the functor

$$\mathrm{w}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}^{op} \to \mathrm{CAT}_1, \quad \mathcal{E} \mapsto \mathrm{Fun}^q(\mathcal{E}),$$

where $\mathrm{Fun}^q(\mathcal{E}) \subset \mathrm{Fun}(\mathcal{E}^{op},\mathcal{A}b)$ denotes the full subcategory spanned by quadratic left exact functors. The unstraightening $\mathcal{C}\mathrm{at}_1^{h,w} := \int_{\mathrm{w}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}} \mathrm{Fun}^q$ has objects pairs $(\mathcal{E},Q,w)$ for $(\mathcal{E},w)$ an exact category with weak equivalences and Q a quadratic functor on $\mathcal{E}$, and maps $(F,\varphi_q) : (\mathcal{E},Q,w) \to (\mathcal{E}',Q',w')$ the data of an exact functor $F : (\mathcal{E},w) \to (\mathcal{E}',w')$ along with a natural transformation $\varphi_q : Q \to Q'\circ F^{op}$. Define $\mathrm{w}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_1^{h,w}$ to be the (non-full) subcategory consisting of tuples $(\mathcal{E},Q,w)$ and maps (F,φ_q) which together satisfy the following:

(i) The polarisation $B_Q(-,-)$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{E}}(-,\mathbb{D}(-))$ for some duality $(\mathbb{D},\eta)$ on $\mathcal{E}$, and the associated double dual identification $1 \to \mathbb{D}\circ\mathbb{D}^{op}$ is a natural weak equivalence.

(ii) The natural transformation $\varphi_q : Q \to Q'\circ F^{op}$ induces a duality compatibility φ as in Remark 2.1.8 which is a natural weak equivalence, and such that for each $x \in \mathcal{E}$, the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_{f(x)}} & \mathbb{D}^2_{Q'}(F(x)) \\ \downarrow{\scriptstyle f(\eta_x)} & & \downarrow{\scriptstyle \mathbb{D}_{Q'}(\varphi_x)} \\ F(\mathbb{D}^2_Q(x)) & \xrightarrow{\varphi_{\mathbb{D}_Q(x)}} & \mathbb{D}_{Q'}(F(\mathbb{D}_Q(x))) \end{array}$$

commutes.

Restricting along the full inclusion $\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \subset \mathrm{w}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$, $\mathcal{E} \mapsto (\mathcal{E},\mathrm{iso})$ we likewise construct the category $\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_1^{h,w}$ of exact form categories and nonsingular form functors; we likewise write $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_1^{h,w}$ for the pullback along the forgetful functor $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \subset \mathrm{w}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$, for $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$ the subcategory of (small) exact categories with weak equivalences and complicial structures, and exact functors between them (see below).

Write $\mathscr{C}_{\mathbb{Z}}$ for the category of bounded chain complexes of finitely generated free abelian groups. Equipped with the degreewise-split exact structure, the class of chain homotopy equivalences, and the duality $X \mapsto [X, \mathbb{1}]$ coming from the closed symmetric monoidal structure furnished by the tensor product of bounded chain complexes, $\mathscr{C}_{\mathbb{Z}}$ carries the canonical structure of an exact category with weak equivalences and duality $(\mathscr{C}_{\mathbb{Z}}, \mathbf{ch.htp.}, [-, \mathbb{1}], \mathrm{can})$, for can_X the adjunct of the evaluation morphism $X \otimes [X, \mathbb{1}] \to \mathbb{1}$ (see Appendix A.3). The Dwyer-Kan localisation $L_{\mathrm{ch.htp.}}(\mathscr{C}_{\mathbb{Z}})$ at the chain homotopy equivalences is canonically equivalent to the perfect derived ∞-category $\mathrm{Perf}(\mathbb{Z})$.

Recall that a **complicial exact category** (see Appendix A.3) is an exact category equipped with a bi-exact action

$$\otimes : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$$

which is associative and unital in that the usual diagrams commute (see [Gra76]). An exact category with weak equivalences and duality $(\mathcal{E}, w, \mathbb{D}, \eta)$ has the structure of a complicial exact category with weak equivalences and duality if $\mathcal{E}$ has a complicial structure such that the action of $\mathscr{C}_{\mathbb{Z}}$ preserves weak equivalences in each variable, i.e. for $f : A \to B$ a chain homotopy equivalence in $\mathscr{C}_{\mathbb{Z}}$ and $g : X \to Y$ in $w\mathcal{E}$, the map

$$f \otimes g : A \otimes X \to B \otimes Y$$

is a weak equivalence; and if $\otimes$ refines to a nonsingular form functor between exact categories with duality, i.e. that we have a natural *isomorphism* φ rendering the diagram

$$\begin{array}{ccc} \mathscr{C}_{\mathbb{Z}}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} & \xrightarrow{[-,\mathbb{1}]\times\mathbb{D}} & \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \\ \big\downarrow{\scriptstyle \otimes^{\mathrm{op}}} & \overset{\varphi}{\swarrow} & \big\downarrow{\scriptstyle \otimes} \\ \mathcal{E}^{\mathrm{op}} & \xrightarrow{\mathbb{D}} & \mathcal{E} \end{array}$$

commutative.

Remark 2.1.10. We will often abuse notation and write $\mathcal{E}$ for an exact category with a complicial structure, leaving the tensor product and higher coherences implicit.

Remark 2.1.11. Write C for the complex $(0 \to \mathbb{Z} = \mathbb{Z} \to 0)$ concentrated in degrees $[0, 1]$, sitting in the congression $\mathbb{1} \rightarrowtail C \twoheadrightarrow T$, and $P := [C, \mathbb{1}]$, sitting in the shifted congression $\Omega \rightarrowtail P \twoheadrightarrow \mathbb{1}$. Then the tuple $(C \otimes -, P \otimes -, \varphi)$ is the data of a strong symmetric cone on $\mathcal{E}$ in the sense of [Sch24b, Def. 9.1], since each of C, P is Frobenius contractible in $\mathscr{C}_{\mathbb{Z}}$.

Warning 2.1.12. The action of $\mathscr{C}_{\mathbb{Z}}$ on $\mathcal{E}$ in the definition of a complicial exact form category is *not* required to promote to a form functor

$$(\mathscr{C}_{\mathbb{Z}}, Q^s, [-, \mathbb{1}], \mathrm{can}) \otimes (\mathcal{E}, Q, \mathbb{D}, \eta) \to (\mathcal{E}, Q, \mathbb{D}, \eta),$$

where $\otimes$ denotes the tensor product of form categories of [Sch21, Def. 2.34]; that is, we require no compatibility between the functor Q and the action of $\mathscr{C}_{\mathbb{Z}}$.

Example 2.1.13. For $(\mathcal{E}, \mathbb{D}, \eta)$ an exact category with duality, the category of bounded chain complexes $\mathrm{Ch}_b(\mathcal{E})$ canonically acquires the structure of a complicial exact category with weak equivalences (the quasi-isomorphisms) and duality; see Appendix A.3. If $\mathcal{E}$ is complicial exact with weak equivalences and I is a small category, the functor category $\mathrm{Fun}(I, \mathcal{E})$ carries a canonical (pointwise) structure of a complicial exact category with weak equivalences. Finally, the opposite of a complicial exact category with weak equivalences canonically has the structure of a complicial exact category with weak equivalences (see (A.4)).

A complicial exact category $\mathcal{E}$ gives rise to an internal class of weak equivalences, the **Frobenius equivalences** (the complicial analogue of chain homotopy equivalences), and an associated **Frobenius exact structure** (see Remark A.3.6) $\mathcal{E}_{\mathrm{Frob}}$ rendering the pair $(\mathcal{E}_{\mathrm{Frob}}, w_{\mathrm{Frob}})$ an exact category with weak equivalences. Note that in general the pair $(\mathcal{E}, w_{\mathrm{Frob}})$ is **not** the data of an exact category with weak equivalences. Any class w of weak equivalences on $\mathcal{E}$ contains the Frobenius equivalences (A.3.10). There are exact inclusions

$$(\mathcal{E}_{\mathrm{Frob}}, w_{\mathrm{Frob}}) \to (\mathcal{E}_{\mathrm{Frob}}, w) \to (\mathcal{E}, w)$$

with underlying functors $\mathrm{id}_{\mathcal{E}}$, presenting the localisations

$$L_{w_{\mathrm{Frob}}}(\mathcal{E}_{\mathrm{Frob}}) =: L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to L_w(\mathcal{E}_{\mathrm{Frob}}) \to L_w(\mathcal{E}),$$

where the second localisation is an equivalence since these are localisations at the same class of weak equivalences, and moreover, being induced by the exact functor $(\mathcal{E}_{\mathrm{Frob}}, w) \to (\mathcal{E}, w)$ is an exact functor of stable ∞-categories by Proposition A.3.13. We may abuse notation by writing $L_{\mathrm{Frob}}(\mathcal{E}) := L_{w_{\mathrm{Frob}}}(\mathcal{E}_{\mathrm{Frob}})$.

2.2 From exact categories to chain complexes

Given an exact form category $(\mathcal{E}, Q, \mathbb{D}, \eta)$, there is a fully faithful exact form functor [Sch24b]

$$(\mathcal{E}, Q, \mathbb{D}, \eta, \mathbf{iso}) \to (\mathrm{Ch}_b(\mathcal{E}), Q_{\mathrm{ch}}, \mathbb{D}, \eta, \mathbf{qis}), \tag{2.3}$$

of exact form categories with weak equivalences. Here we abusively write $(\mathbb{D}, \eta)$ for the canonical extensions of the duality on $\mathcal{E}$ to $\mathrm{Ch}_b(\mathcal{E})$ (see Appendix A.3), and Q_{ch} for the quadratic left-exact functor defined on a chain complex X by

$$Q_{\mathrm{ch}}(X) := \{(\xi, \varphi) \mid \varphi \in \mathrm{Hom}_{\mathrm{Ch}_b(\mathcal{E})}(X, \mathbb{D}(X))^{C_2}, \xi \in Q(X[0]), d_1^\bullet(\xi) = 0, \rho_{X[0]}(\xi) = \varphi_0\}; \tag{2.4}$$

see [Sch24b, Def. 10.2]. The following Gillet-Waldhausen-type theorem is [Sch24b, Thm. 10.5].

Theorem 2.2.1. *For $\mathcal{E}$ an exact form category, the form functor (2.3) induces a natural weak equivalence of Grothendieck-Witt spaces*

$$\mathcal{GW}(\mathcal{E}, Q) \to \mathcal{GW}(\mathrm{Ch}_b(\mathcal{E}), Q_{\mathrm{ch}}, \mathbf{qis}).$$

Many of the categorical properties of $\mathrm{Ch}_b(\mathcal{E})$ (simplicial enrichment, stability of the localisation at the chain homotopy equivalences) ultimately derive from its complicial structure. We couch the discussion below in terms of complicial exact categories with weak equivalences, showing below that derived functors of presheaves on these admit explicit models. In turn, complicial exact form categories with weak equivalences are tractable 1-categorical models for Poincaré categories.

2.3 Grothendieck-Witt theory of exact categories

In this section we recall the construction of the Grothendieck-Witt space associated to an exact form category with weak equivalences; see [Sch21, §6], [Sch24b, §7] for details.

To an exact form category with weak equivalences $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$, we may associate the category of quadratic spaces $w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w)$. This is the Grothendieck construction on the functor Q_{nd} of Remark 3.2.1,

$$Q_{nd} : w\mathcal{E}^{op} \to \mathcal{S}\mathrm{et} \subset \mathcal{C}\mathrm{at}, x \mapsto \{\xi \in Q(x) \mid \rho_x(\xi) \in w\mathcal{E}\},$$

classifying the right fibration $w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w) \to w\mathcal{E}$, $(x, q) \mapsto x$. Explicitly, objects are pairs (x, ξ) for x an object of $\mathcal{E}$ and $\xi \in Q(x)$ such that the associated symmetric form $\rho_x(\xi) \in \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))$ is a weak equivalence, and maps $(x, \xi) \to (y, \zeta)$ are weak equivalences $f : x \to y$ in $\mathcal{E}$ satisfying $f^{\bullet}(\zeta) = \xi$.

Suppose now given a poset $(\mathcal{P}, \leq)$ with strict duality $\mathcal{P}^{op} \to \mathcal{P}$, $i \mapsto \bar{i}$. For an exact form category with weak equivalences $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$, the category $\mathrm{Fun}(\mathcal{P}, \mathcal{E})$ acquires the structure of an exact form category with weak equivalences as follows. The dual of a functor $X : \mathcal{P} \to \mathcal{E}$, $i \mapsto X_i$ is the conjugated functor

$$\mathbb{D}_{\mathcal{P}}(X) : i \mapsto \mathbb{D}(X_{\bar{i}}),$$

with double dual identification given pointwise by η. A natural transformation $X \to Y$ is a weak equivalence if it is so pointwise. A quadratic form on a functor $X : \mathcal{P} \to \mathcal{E}$ is the data of a pair $((\xi_i)_{i\leq\bar{i}}, \varphi)$, for $(\xi_i)_{i\leq\bar{i}} \in \prod_{i\leq\bar{i}} Q(X_i)$ and $\varphi : X \to \mathbb{D}_{\mathcal{P}}(X)$ a symmetric form on X, satisfying $X^{\bullet}_{i\leq j}(\xi_j) = \xi_i$ for each pair i, j satisfying $i \leq j \leq \bar{j} \leq \bar{i}$, and $\rho_{X_i}(\xi_i) = \varphi_{\bar{i}} \circ X_{i\leq\bar{i}}$. The set $Q_{\mathcal{P}}(X)$ of such forms is an abelian group with $(\{\xi_i\}_{i\leq\bar{i}}, \varphi) + (\{\eta_j\}_{j\leq\bar{j}}, \psi) = (\{\xi_i + \zeta_i\}_{i\leq\bar{i}}, \varphi + \psi)$. Informally, a form on X is a compatible family of pointwise forms indexed by the subposet of elements $i \in \mathcal{P}$ with $i \leq \bar{i}$, and a symmetric form on X compatible with the family of symmetric forms coming from the ξ_i; we shall see below that in the case $\mathcal{P} = \mathrm{Ar}([n])$, the induced structure of an exact form category on $S_n(\mathcal{E})$ is such that the map φ is determined uniquely by the ξ_i. Functoriality is inherited from that of Q on $\mathcal{E}$: for $\theta : X \to Y$ a natural transformation, we define

$$Q_{\mathcal{P}}(Y) \to Q_{\mathcal{P}}(X), \quad (\{\xi_i\}_{i\leq\bar{i}}, \varphi) \mapsto (\{\theta^{\bullet}(\xi_i)\}_{i\leq\bar{i}}, \mathbb{D}(\theta)\varphi\theta).$$

Associated to each functor X is the C_2-Mackey functor

$$\mathrm{Nat}(X, \mathbb{D}_{\mathcal{P}}(X)) \xrightarrow{\tau_X} Q_{\mathcal{P}}(X) \xrightarrow{\rho_X} \mathrm{Nat}(X, \mathbb{D}_{\mathcal{P}}(X))$$
$$\tau_X : \varphi \mapsto (\{\tau(\varphi_{\bar{i}} \circ X_{i\leq\bar{i}})\}_{i\leq\bar{i}}, \varphi + \sigma(\varphi)),$$
$$\rho_X : (\{\xi_i\}_{i\leq\bar{i}}, \varphi) \mapsto \varphi,$$

from which one may check that $Q_{\mathcal{P}}$ is quadratic by the criteria of [Sch21, Lem. A.10(ii)].

Recall that the category $\mathrm{Ar}([n])$ has a unique strict duality $(i \leq j) \mapsto (n-j \leq n-i)$; for $\mathcal{E}$ as above, the category $\mathrm{Fun}(\mathrm{Ar}([n]), \mathcal{E})$ thus inherits the structure of an exact form category with weak equivalences. To ease notation, we write $X^{i\leq j}_{k\leq l}$ for the image under X of the map $(i \leq j) \leq (k \leq l)$, and $Q_n := Q_{\mathrm{Ar}([n])}$. Restricting to the subcategory of functors $X : \mathrm{Ar}([n]) \to \mathcal{E}$ such that $X_{i\leq i} \cong 0$, and such that for each $0 \leq i \leq j \leq k \leq n$ the sequence

$$X_{i\leq j} \rightarrowtail X_{i\leq k} \twoheadrightarrow X_{j\leq k}$$

is a congression in $\mathcal{E}$, we obtain an exact form category with weak equivalences $(S_n(\mathcal{E}), Q, \mathbb{D}, \eta, w)$. Denote by $\mathcal{P}_n \subset \mathrm{Ar}([n])$ the subposet of arrows $(i \leq j)$ satisfying

$$(i \leq j) \leq \overline{(i \leq j)} = (n-j \leq n-i),$$

i.e. with $i + j \leq n$. A quadratic form on $X \in S_n(\mathcal{E})$ is a pair $(\{\xi_{i\leq j}\}_{i+j\leq n}, \varphi)$, with the family $(\xi_{i\leq j})$ an object in the limit $\lim_{(i\leq j)\in\mathcal{P}_n^{op}} Q$, satisfying

$$\rho_{X_{i\leq j}}(\xi_{i\leq j}) = \varphi_{n-j\leq n-i} \circ X^{i\leq j}_{(n-j\leq n-i)}.$$

In particular we have $\rho_{X_{0\leq n}}(\xi_{0\leq n}) = \varphi_{0\leq n}$, and as noted in [HS25, Rem. 8.2.5(i)], φ is determined by the $\xi_{i\leq j}$: for each $0 \leq i \leq n$ we have a map of congressions

$$\begin{array}{ccccc} X_{0\leq i} & \xrightarrow{X^{0\leq i}_{0\leq n}} & X_{0\leq n} & \xrightarrow{X^{0\leq n}_{i\leq n}} & X_{i\leq n} \\ \downarrow \varphi_{0\leq i} & & \downarrow \varphi_{0\leq n} & & \downarrow \varphi_{i\leq n} \\ \mathbb{D}(X_{n-i\leq n}) & \xrightarrow{\mathbb{D}(X^{0\leq n}_{n-i\leq n})} & \mathbb{D}(X_{0\leq n}) & \xrightarrow{\mathbb{D}(X^{0\leq n-i}_{0\leq n})} & \mathbb{D}(X_{0\leq n-i}), \end{array}$$

from which we see that $\varphi_{0\leq i}$ and $\varphi_{i\leq n}$ are determined by $\varphi_{0\leq n}$, and repeating this with the map of congressions associated with $X_{0\leq i} \rightarrowtail X_{0\leq j} \twoheadrightarrow X_{i\leq j}$, similarly for $\varphi_{i\leq j}$. Accordingly, we have an injection

$$Q_n(X) \hookrightarrow \lim_{(i\leq j)\in\mathcal{P}_n^{op}} Q \circ X^{op}, \quad ((\xi_i)_i, \varphi) \mapsto (\xi_i)_i.$$

Construction 2.3.1. Contravariant naturality of the $S_\bullet$-construction yields a simplicial exact category $S_\bullet(\mathcal{E})$. The simplicial structure maps of $S_\bullet(\mathcal{E})$ are however not compatible with the dualities, but are if we consider the edgewise subdivision $S^e_\bullet(\mathcal{E})$, obtained by precomposing with the subdivision $\mathrm{sd} : \Delta^{op} \to \Delta^{op}$, $[n] \mapsto [n]^{op} \star [n] \cong [2n+1]$ (this has the effect of symmetrising faces and degeneracies). We thus obtain a simplicial exact form category with weak equivalences, and a functor

$$S_\bullet : \mathrm{w}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \to \Delta^{op}\mathrm{w}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}, \quad (\mathcal{E}, Q, \mathbb{D}, \eta, w) \mapsto (S^e_\bullet(\mathcal{E}), Q^e_\bullet, \mathbb{D}^e_\bullet, \eta^e_\bullet, w).$$

Postcomposing with $\mathrm{w}\mathcal{Q}\mathrm{uad}$ and taking nerves, we have a simplicial space

$$[p] \mapsto |\mathrm{w}\mathcal{Q}\mathrm{uad}(S_{2p+1}(\mathcal{E}), Q_{2p+1}, w)|$$

The forgetful functors

$$\mathrm{w}\mathcal{Q}\mathrm{uad}(S_n\mathcal{E}, Q_n, w) \to wS_n(\mathcal{E})$$

induce a map of simplicial spaces

$$[p] \mapsto (|\mathrm{w}\mathcal{Q}\mathrm{uad}(S_{2p+1}(\mathcal{E}), Q_{2p+1}, w)| \to |wS_{2p+1}(\mathcal{E})| \to |wS_p(\mathcal{E})|),$$

where the last map is induced levelwise by the natural transformation $\mathrm{id}_\Delta \Rightarrow \mathrm{sd}$, $[n] \subset [n]^{op} \star [n]$, $i \mapsto i$. The Grothendieck-Witt space [Sch21, Def. 6.3] is defined to be the homotopy fibre of the corresponding map of geometric realisations

$$\mathcal{GW}(\mathcal{E}, Q, w) \to |\mathrm{w}\mathcal{Q}\mathrm{uad}(S^e_\bullet(\mathcal{E}), Q^e_\bullet, w)| \to |wS_\bullet(\mathcal{E})|.$$

Remark 2.3.2. The constructions in [Sch21] and [Sch24b] make use of the functor

$$\mathrm{sd}_0 : \Delta \to \Delta, \quad [n] \mapsto [n]^{op} \star [n],$$

while the authors of [CDH+II] work with the $\mathcal{Q}_\bullet$-construction which under the equivalence $\mathcal{Q}_n(\mathcal{C}) \simeq S^e_n(\mathcal{E})$ corresponds to the subdivision

$$\mathrm{sd}_1 : \Delta \to \Delta, \quad [n] \mapsto [n] \star [n]^{op}.$$

Here we write $(-)^{op} : \Delta^{op} \to \Delta^{op}$ for the involution $[n] \mapsto [n]^{op}, \theta \mapsto \theta^{op}$, where $[n]^{op} = (n' < (n-1)' < \cdots < 1' < 0')$ and $\theta^{op} : [m]^{op} \to [n]^{op}$ is $i' \mapsto \theta(i)'$. Given a map $\theta : [m] \to [n]$, $sd_0(\theta)$ (resp. $sd_1(\theta)$) is the map $\theta^{op} \star \theta$ (resp. $\theta \star \theta^{op}$). There is an identification of simplicial spaces

$$\left([n] \mapsto \left|wQuad(S_{sd_0[n]}(\mathcal{E}), Q_{sd_0[n]})\right|\right) = \left([n] \mapsto \left|wQuad(S_{sd_1[n]}(\mathcal{E}), Q_{sd_1[n]})\right|\right)^{op},$$

where the opposite of a simplicial space X is given by precomposition with $(-)^{op}$. Accordingly, since $(-)^{op}$ is cofinal (see for instance [Bar13, §2]), we have equivalences

$$\begin{array}{ccc} |wQuad(sd_0^*(S_\bullet(\mathcal{E}), Q_\bullet))| & \longrightarrow & |wS_\bullet(\mathcal{E})| \\ \downarrow \wr & & \downarrow \wr \\ |wQuad(sd_1^*(S_\bullet(\mathcal{E}), Q_\bullet))| & \longrightarrow & |wS_\bullet(\mathcal{E})^{op}| \end{array}$$

and an induced equivalence on fibres.

We shall see in §5.1 that the Grothendieck-Witt space admits a canonical delooping to a (generally nonconnective) spectrum.

2.4 Hermitian and Poincaré ∞-categories

In this brief interlude we recall some of the technical background for Grothendieck-Witt theory in the higher-categorical setting; for a detailed exposition, see [CDH+I, §§1-2]. Recall [HA, Def. 1.1.1.9] that an ∞-category is stable if it is pointed (i.e., has a zero object), admits all fibres and cofibres, and has the property that a nullcomposite sequence

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array}$$

is a fibre sequence if and only if it is a cofibre sequence. Fix a small stable ∞-category $\mathcal{C}$. To any reduced functor $Ϙ : \mathcal{C}^{op} \to \mathcal{S}p$ we may associate its polarisation

$$B_Ϙ(x, y) := \mathrm{fib}(Ϙ(x \oplus y) \to Ϙ(x) \oplus Ϙ(y));$$

this assignment is functorial in $Ϙ$, and exhibits $B_Ϙ$ as the universal bi-reduced[4] replacement of the bifunctor $\mathcal{C}^{op} \times \mathcal{C}^{op} \xrightarrow{\oplus} \mathcal{C}^{op} \xrightarrow{Ϙ} \mathcal{S}p$ (see [CDH+I, §1]) in the following sense: write $\mathrm{BiFun}_*(\mathcal{C}^{op}; \mathcal{S}p)$ for the ∞-category of bi-reduced functors $\mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$. Then we have natural transformations

$$\Delta^* B_Ϙ \Rightarrow Ϙ \Rightarrow \Delta^* B_Ϙ, \tag{2.5}$$

which are counit and unit in adjunctions exhibiting $Ϙ \mapsto B_Ϙ$ as respectively right and left adjoint to restriction along the diagonal $\Delta^* : \mathrm{BiFun}_*(\mathcal{C}^{op}; \mathcal{S}p) \to \mathrm{Fun}_*(\mathcal{C}^{op}, \mathcal{S}p)$. For $Ϙ$ reduced, $B_Ϙ$ refines to a C_2-homotopy fixed point in $\mathrm{BiFun}_*(\mathcal{C}^{op}, \mathcal{S}p)$, where C_2 acts by flipping the input variables; we call such a functor symmetric. By [CDH+I, Lem. 1.1.10], the maps (2.5) refine to C_2-equivariant transformations, where C_2 acts trivially on $\mathrm{Fun}_*(\mathcal{C}^{op}, \mathcal{S}p)$, and hence to maps

$$(\Delta^* B_Ϙ)_{hC_2} \Rightarrow Ϙ \Rightarrow (\Delta^* B_Ϙ)^{hC_2}.$$

[4] A functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ is bi-reduced if for each $x \in \mathcal{C}$, each restriction $B(x, -), B(-, x) : \mathcal{C}^{op} \to \mathcal{S}p$ is reduced.

Call a bi-reduced functor $B_{\mathfrak{Q}} : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{S}p$ bilinear if it is $(1,1)$-excisive, i.e. 1-excisive in each variable. Then by [CDH+I, Prop. 1.1.13], a reduced functor $\mathfrak{Q} : \mathcal{C}^{op} \to \mathcal{S}p$ is 2-excisive if and only if its polarisation $B_{\mathfrak{Q}}$ is bilinear, and if the functor

$$\Lambda_{\mathfrak{Q}} : \mathcal{C}^{op} \to \mathcal{S}p, \quad x \mapsto \mathrm{fib}\left(\mathfrak{Q}(x) \to B_{\mathfrak{Q}}(x,x)^{hC_2}\right)$$

is 1-excisive. We use also the term quadratic to mean reduced and 2-excisive, and call the pair $(\mathcal{C}, \mathfrak{Q})$ a hermitian ∞-category. Analogously to the case of form categories, one defines a notion of quadratic functors on $\mathcal{C}$ satisfying an additional nondegeneracy condition: call a quadratic functor $\mathfrak{Q}$ nondegenerate if there is a natural equivalence

$$B_{\mathfrak{Q}}(x,y) \simeq \hom_{\mathcal{C}}(x, \mathbb{D}_{\mathfrak{Q}}(y))$$

for some (necessarily essentially unique) duality $\mathbb{D}_{\mathfrak{Q}} : \mathcal{C}^{op} \to \mathcal{C}$. The equivalences

$$\hom_{\mathcal{C}}(x, \mathbb{D}_{\mathfrak{Q}}(y)) \simeq B_{\mathfrak{Q}}(x,y) \simeq B_{\mathfrak{Q}}(y,x) \simeq \hom_{\mathcal{C}}(y, \mathbb{D}_{\mathfrak{Q}}(x)) \simeq \hom_{\mathcal{C}^{op}}(\mathbb{D}_{\mathfrak{Q}}^{op}(x), y)$$

exhibit $\mathbb{D}_{\mathfrak{Q}}$ as self-adjoint, and we call $B_{\mathfrak{Q}}$ perfect if the unit (and hence also counit) $\mathrm{ev} : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathbb{D}_{\mathfrak{Q}} \mathbb{D}_{\mathfrak{Q}}^{op}$ of this adjunction is a natural equivalence.

Definition 2.4.1 ([CDH+I, Def. 1.2.8])**.** A hermitian ∞-category $(\mathcal{C}, \mathfrak{Q})$ is Poincaré if $B_{\mathfrak{Q}}$ is nondegenerate and perfect; this is in some sense the higher-categorical generalisation of a form category.

Write $\mathrm{Fun}^q(\mathcal{C})$ for the subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}p)$ spanned by reduced 2-excisive functors, and write $\mathcal{C}at_\infty^h$ for the cartesian unstraightening of

$$(\mathcal{C}at_\infty^{st})^{op} \to \mathcal{C}AT_\infty, \quad \mathcal{C} \mapsto \mathrm{Fun}^q(\mathcal{C}),$$

the (large) ∞-category of small hermitian ∞-categories. A map $(\mathcal{C}, \mathfrak{Q}) \to (\mathcal{C}', \mathfrak{Q}')$ in $\mathcal{C}at_\infty^h$ is a pair (f, η), for $f : \mathcal{C} \to \mathcal{C}'$ an exact functor, and $\eta : \mathfrak{Q} \Rightarrow \mathfrak{Q}' \circ f^{op}$. Such an η induces a map (see [CDH+I, Lem. 1.2.4])

$$\theta_{f,\eta} : f\,\mathbb{D}_{\mathfrak{Q}} \Rightarrow \mathbb{D}_{\mathfrak{Q}'} \circ f^{op},$$

and we call the pair (f, η) duality preserving if $\theta_{f,\eta}$ is an equivalence. Denote by $\mathcal{C}at_\infty^p \subset \mathcal{C}at_\infty^h$ the subcategory spanned by Poincaré ∞-categories and duality-preserving functors.

2.5 Grothendieck-Witt theory for stable categories

Fix a Poincaré ∞-category $(\mathcal{C}, \mathfrak{Q})$; we have a presheaf of spaces $\Omega^\infty \mathfrak{Q}$ on $\mathcal{C}$, associating to each $x \in \mathcal{C}$ the space of forms on x, the cartesian unstraightening of which is denoted $\mathrm{He}(\mathcal{C}, \mathfrak{Q})$. Call a form $\xi \in \Omega^\infty \mathfrak{Q}(x)$ *nondegenerate* if the symmetric form associated via the natural map $\mathfrak{Q}(x) \to B_{\mathfrak{Q}}(x,x)$ is an equivalence. There is a subfunctor inclusion

$$\mathfrak{Q}_{nd} \subset \Omega^\infty \mathfrak{Q} : \mathcal{C}^{op} \to \mathcal{S}, \tag{2.6}$$

with $\mathfrak{Q}_{nd}(x)$ the subspace spanned by the nondegenerate forms on x, i.e. the pullback

$$\begin{array}{ccc} \mathfrak{Q}_{nd}(x) & \longrightarrow & \Omega^\infty \mathfrak{Q}(x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}^{\simeq}}(x, \mathbb{D}(x)) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, \mathbb{D}(x)). \end{array} \tag{2.7}$$

The inclusion $\Omega_{nd} \subset \Omega^\infty \Omega$ gives rise to a map of right fibrations over $\mathcal{C}$

$$\int_{\mathcal{C}} \Omega_{nd} \to \mathrm{He}(\mathcal{C}, \Omega),$$

and pulling back along the inclusion $\mathcal{C}^{\simeq} \subset \mathcal{C}$, naturality of the straightening-unstraightening equivalence yields a square in $\mathcal{C}\mathrm{at}_\infty$

$$\begin{array}{ccc} \int_{\mathcal{C}^{\simeq}} \Omega_{nd} & \longrightarrow & \mathrm{He}(\mathcal{C}^{\simeq}, \Omega) \\ \downarrow & & \downarrow \\ \int_{\mathcal{C}} \Omega_{nd} & \longrightarrow & \mathrm{He}(\mathcal{C}, \Omega) \end{array} \tag{2.8}$$

commuting up to canonical equivalence.

Lemma 2.5.1. *Suppose given a functor* $F : \mathcal{D}^{op} \to \mathcal{S}$ *classifying the right fibration* $\int_{\mathcal{D}} F \to \mathcal{D}$*, and denote by* $\iota : \mathcal{D}^{\simeq} \subset \mathcal{D}$ *the inclusion of the groupoid core. Then there is a natural equivalence of spaces*

$$\left(\int_{\mathcal{D}} F\right)^{\simeq} \xrightarrow{\simeq} \iota^* \int_{\mathcal{D}} F = \int_{\mathcal{D}^{\simeq}} \iota^* F.$$

Proof. That $\iota^* \int_{\mathcal{D}} F$ is a space follows from the fact that a right fibration over a Kan complex is a Kan fibration. The pullback square

$$\begin{array}{ccc} \iota^* \int_{\mathcal{D}} F & \longrightarrow & \int_{\mathcal{D}} F \\ \downarrow & & \downarrow \\ \mathcal{D}^{\simeq} & \xhookrightarrow{\iota} & \mathcal{D} \end{array}$$

induces an essentially unique map $\left(\int_{\mathcal{D}} F\right)^{\simeq} \to \iota^* \int_{\mathcal{D}} F$, and given a space X, we observe that this induces a natural equivalence

$$\begin{aligned} \mathrm{Map}_{\mathcal{S}}\Big(X, \Big(\int_{\mathcal{D}} F\Big)^{\simeq}\Big) &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}\Big(X, \int_{\mathcal{D}} F\Big) \\ &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}\Big(X, \int_{\mathcal{D}} F\Big) \times_{\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(X, \mathcal{D})} \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(X, \mathcal{D}^{\simeq}) \\ &\simeq \mathrm{Map}_{\mathcal{S}}\Big(X, \iota^* \int_{\mathcal{D}} F\Big), \end{aligned}$$

since the composite $X \to \int_{\mathcal{D}} F \to \mathcal{D}$ necessarily factors through $\iota : \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D}$. □

The upper-right corner in (2.8) thus identifies with the maximal subgroupoid $\mathrm{Fm}(\mathcal{C}, \Omega) \subset \mathrm{He}(\mathcal{C}, \Omega)$ of [CDH+I, Def. 2.1.1], and the upper left with the subgroupoid $\mathrm{Pn}(\mathcal{C}, \Omega) \subset \mathrm{Fm}(\mathcal{C}, \Omega)$ spanned by pairs (x, ξ), for $\xi \in \Omega^\infty \Omega(x)$ nondegenerate. Pn assembles into a functor $\mathcal{C}\mathrm{at}^p_\infty \to \mathcal{S}$ (note that the restriction to duality-preserving functors in $\mathcal{C}\mathrm{at}^p_\infty$ is necessary for this) which is moreover corepresented by a compact object ([CDH+I, Prop. 4.1.3, Prop. 6.1.8]) and preserves limits and filtered colimits.
For an ∞-category K, denote by $\mathrm{TwAr}(K)$ the twisted arrow category of K, with n-simplices

$$\mathrm{TwAr}_n(K) = \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n \star (\Delta^n)^{op}, K).$$

We may equip $\mathrm{Fun}(\mathrm{TwAr}(K), \mathcal{C})$ with a quadratic functor

$$\Omega_K = \lim_{\mathrm{TwAr}(K)^{op}} \Omega,$$

and restricting this to the subcategory spanned by functors F for which the squares

$$\begin{array}{ccc} F(i \to l) & \longrightarrow & F(i \to k) \\ \downarrow & & \downarrow \\ F(j \to l) & \longrightarrow & F(j \to k) \end{array}$$

are exact in $\mathcal{C}$ for each map $\Delta^3 \to K$, $i \to j \to k \to l$, we obtain a hermitian ∞-category $(\mathcal{Q}_K(\mathcal{C}, \Omega), \Omega_K)$ which is in fact Poincaré by [CDH+II, Lem. 2.2.6]. Restricting along the inclusion $\Delta \subset \mathcal{C}\mathrm{at}_\infty$, we obtain a simplicial Poincaré ∞-category, denoted $(\mathcal{Q}_\bullet(\mathcal{C}), \Omega_n)$. As in the classical setting, there is a forgetful functor

$$\mathrm{Pn}(\mathcal{Q}_n(\mathcal{C}), \Omega_n) \to \mathcal{Q}_n(\mathcal{C})^\simeq,$$

and the Grothendieck-Witt space can be defined [CDH+II, Cor. 4.1.7] as the fibre

$$\mathcal{GW}(\mathcal{C}, \Omega) \to |\mathrm{Pn}(\mathcal{Q}_\bullet(\mathcal{C}), \Omega_\bullet)| \to |\mathcal{Q}_\bullet(\mathcal{C})^\simeq|.$$

Remark 2.5.2. As in [CDH+II, App. B], it will be convenient to reformulate the above constructions in terms of the hermitian $S^e_\bullet$-construction: there is a functor

$$\iota_n : \mathrm{TwAr}(\Delta^n) \to \mathrm{Ar}(\Delta^n \star (\Delta^n)^{\mathrm{op}})$$

sending $(i \leq j)$ to $(i \leq j')$, where we write the vertices of $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ as

$$(0 < \cdots < (n-1) < n < n' < (n-1)' < \cdots < 0'),$$

which is well known to induce a levelwise equivalence of simplicial stable ∞-categories

$$\iota_n^* : S^e_\bullet(\mathcal{C}) \to \mathcal{Q}_\bullet(\mathcal{C}).$$

We equip $S^e_n(\mathcal{C}) = S_{2n+1}(\mathcal{C})$ with a hermitian structure as follows. Write $\mathcal{P}_n \subset \mathrm{Ar}(\Delta^n)$ for the subposet on pairs $(i \leq j)$ such that $i + j \leq n$. Under the isomorphism

$$\mathrm{Ar}(\Delta^{2n+1}) \cong \mathrm{Ar}(\Delta^n \star (\Delta^n)^{\mathrm{op}}),$$

$\mathcal{P}_{2n+1}$ corresponds to the subposet on pairs $i \leq j$ or $k \leq l'$, for $0 \leq k \leq l \leq n$. The poset inclusion $\mathrm{TwAr}(\Delta^n) \to \mathrm{Ar}(\Delta^n \star (\Delta^n)^{\mathrm{op}})$ thus factors as

$$\mathrm{TwAr}(\Delta^n) \overset{j_n}{\subset} \mathcal{P}_{2n+1} \subset \mathrm{Ar}(\Delta^n \star (\Delta^n)^{\mathrm{op}})$$

and we claim that j_n is cofinal. Objects of $((i \leq j) \downarrow j_n)$ are maps $(i \leq j) \to (k \leq l')$, for some $0 \leq k \leq l \leq n$, and likewise objects of $((i \leq j') \downarrow j_n)$ are maps $(i \leq j') \to (k \leq l')$. It is easy to see that the former category has initial object $(i \leq j) \to (i \leq n')$, and the latter $(i \leq j') = (i \leq j')$. The claim follows by Quillen's theorem A.

For each $n \geq 0$ we equip $S^e_n(\mathcal{C})$ with quadratic functor

$$X \mapsto \lim_{\mathrm{TwAr}(\Delta^n)^{\mathrm{op}}} \Omega \circ (\iota_n^* X)^{\mathrm{op}} \simeq \lim_{\mathcal{P}_{2n+1}^{\mathrm{op}}} \Omega \circ X^{\mathrm{op}} \tag{2.9}$$

obtaining a Poincaré category $(S^e_n(\mathcal{C}), (\iota_n^*)^* \Omega_n)$ tautologically Poincaré equivalent to $(\mathcal{Q}_n(\mathcal{C}), \Omega_n)$ via ι_n^*.

As in the classical case, the Grothendieck-Witt space of a Poincaré category admits a delooping, which we discuss in 5.2.

3 Deriving form functors along Dwyer-Kan localisations

In §3.1 we compute the (right) derived functor of a quadratic left-exact functor associated to an exact form category with weak equivalences $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ in the sense of Schlichting [Sch21], arriving in §§3.2-3.4 at a canonical Poincaré ∞-category $(L_w(\mathcal{E}), \mathbf{R}Q)$ in the sense of [CDH+I].

3.1 Deriving presheaves on complicial exact categories

Recall from Appendix A.3 that a complicial structure on an exact category is the structure of a module in the symmetric monoidal ∞-category of exact ∞-categories $\mathcal{E}\mathrm{xact}_\infty^\otimes$ [NW25] over the symmetric monoidal exact category $\mathscr{C}_\mathbb{Z}$ of bounded chain complexes of finite free abelian groups; a complicial exact category with weak equivalences and duality $(\mathcal{E}, w)$ is additionally equipped with a wide subcategory of weak equivalence $w \subset \mathcal{E}$, and an exact duality functor $\mathbb{D} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ compatible with the complicial structure. Recall that the identity functor gives rise to an exact inclusion of complicial exact categories with weak equivalences $(\mathcal{E}_{\mathrm{Frob}}, w_{\mathrm{Frob}}) \subset (\mathcal{E}, w)$.

Definition 3.1.1 ([Cis19, Def. 7.4.12, 7.5.7])**.** An ∞-category with weak equivalences and fibrations is a tuple $(\mathcal{C}, w, \mathrm{Fib})$, with $\mathcal{C}$ an ∞-category with final object $*$, $w \subset \mathcal{C}$ a wide subcategory[1] satisfying 2-of-3, and $\mathrm{Fib} \subset \mathcal{C}$ a wide subcategory of fibrations, such that the following properties are satisfied (an object $x \in \mathcal{E}$ is fibrant if the essentially unique map $x \to *$ lies in Fib).

(i) For any cartesian square

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ y' & \longrightarrow & y \end{array}$$

in $\mathcal{C}$ with p a fibration between fibrant objects and y' fibrant, if p belongs to w, so does q.

(ii) For any map $f : x \to y$ with y fibrant, there exists a map $i : x \to x'$ in w and $p : x' \to y$ in Fib such that f is a composition of p and i.

$(\mathcal{C}, \mathrm{Fib}, w)$ is moreover an ∞-category of fibrant objects if each object $x \in \mathcal{C}$ is fibrant.

Lemma 3.1.2. *(The nerve of) any complicial exact category with weak equivalences* $(\mathcal{E}, w)$ *is an ∞-category of fibrant objects upon setting* Fib *to be the class of egressions.*

Proof. Since the sequences $x = x \to 0$ are congressions, each object is fibrant. Given a cartesian square as in (i), if p is an egression then so is q; an egression is then trivial if and only if its kernel is acyclic [Sch24b, Lem. 7.1],

[1] We may occasionally write $w\mathcal{C}$ for w.

and since this is so for p, it follows for q. For (ii), given $f : x \to y$, we note that in the diagram

$$\begin{array}{ccc} x & \xrightarrow{\binom{1}{0}} & x \oplus Py \\ & \overset{f}{\searrow} & \downarrow{\scriptstyle (f\ \pi)} \\ & & y, \end{array}$$

for $\pi : Py \to y$ the path fibration of Appendix A.3, the ingression $x \to x \oplus Py$ has (Frobenius) contractible cokernel Py, and so the former is a (Frobenius) weak equivalence. Now the composite $Py \xrightarrow{\binom{0}{1}} x \oplus Py \to y$ is a fibration, and since the kernel of the latter map exists as the pullback

$$\begin{array}{ccc} \ker((\,f\ \pi\,)) & \longrightarrow & Py \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ x & \xrightarrow{-f} & y, \end{array}$$

$(\,f\ \pi\,)$ is an egression, by Quillen's obscure axiom [Kel90, App. A.1]. □

For $(\mathcal{C}, w, \mathrm{Fib})$ an ∞-category with weak equivalences and fibrations, write $\gamma : \mathcal{C} \to L_w(\mathcal{C})$ for the Dwyer-Kan localisation at the class of weak equivalences. For $x \in \mathcal{C}$ fibrant, the slice category $(\mathcal{C} \downarrow x)$ inherits the structure of an ∞-category with weak equivalences and fibrations in which the weak equivalences resp. fibrations are those maps which are sent to such under the projection $(\mathcal{C} \downarrow x) \to \mathcal{C}$ [Cis19, §7.6.12]. Fibrant objects in this inherited structure are precisely the fibrations $y \twoheadrightarrow x$. We note the following for the record.

Corollary 3.1.3. *Let $(\mathcal{E}, w)$ be a complicial exact category with weak equivalences. Writing $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ for the localisation, the induced functors*

$$\gamma_{x/} : (x \downarrow \mathcal{E}) \to (\gamma(x) \downarrow L_w(\mathcal{E})), \quad \gamma_{/x} : (\mathcal{E} \downarrow x) \to (L_w(\mathcal{E}) \downarrow \gamma(x))$$

are both final and cofinal, for each $x \in \mathcal{E}$.

Proof. By [Cis19, Cor. 7.6.13], the functor $\gamma_{/x}$ induces an equivalence $L_w(\mathcal{E} \downarrow x) \simeq (L_w(\mathcal{E}) \downarrow \gamma(x))$, and is accordingly a Dwyer-Kan localisation. The desired conclusion for $\gamma_{/x}$ then follows since any Dwyer-Kan localisation is both final and cofinal [Cis19, Prop. 7.1.10]; that for $\gamma_{x/}$ follows form the identification $(\mathcal{E} \downarrow x)^{\mathrm{op}} \simeq (x \downarrow \mathcal{E}^{\mathrm{op}})$ and that the opposite of a complicial exact category with weak equivalences is a complicial exact category with weak equivalences. □

Write $\mathrm{sSet} = \mathcal{P}^{\mathrm{Set}}(\mathbb{\Delta})$ for the ordinary category of simplicial sets, and $\mathrm{sSet}^{\mathrm{fin}} \subset \mathrm{sSet}$ for the full subcategory of simplicial sets with finitely many nondegenerate simplices. There is a functor

$$\mathrm{N}\,\mathbb{Z}[-] : \mathrm{sSet}^{\mathrm{fin}} \to \mathscr{C}_{\mathbb{Z}},$$

sending a finite simplicial set to the normalised chain complex on the associated free simplicial abelian group. $\mathrm{N}_n\,\mathbb{Z}[K]$ identifies with the free abelian subgroup $\mathbb{Z}[K_n^{\mathrm{nd}}]$ on the nondegenerate n-simplices of K, with differential

$$\partial_n : \mathrm{N}_n\,\mathbb{Z}[K] \to \mathrm{N}_{n-1}\,\mathbb{Z}[K], \quad \sigma \mapsto \sum_{0 \le i \le n} \begin{cases} (-1)^i d_i^n(\sigma), & d_i^n(\sigma) \text{ nondegenerate}, \\ 0, & \text{else}. \end{cases}$$

The canonical isomorphisms $\mathbb{Z}[K \times L] \cong \mathbb{Z}[K] \otimes \mathbb{Z}[L]$ of simplicial abelian groups, and the Alexander-Whitney and Eilenberg-Zilber maps $\Delta_{A,B} : N(A \otimes B) \to N(A) \otimes N(B)$ and $\nabla_{A,B} : N(A) \otimes N(B) \to N(A \otimes B)$ respectively, for simplicial abelian groups A and B, render $N\,\mathbb{Z}[-]$ an oplax and lax symmetric monoidal functor of symmetric monoidal categories, where the source is given the cartesian symmetric monoidal structure, and the target the symmetric monoidal structure induced by the tensor product of chain complexes (see for instance [May92, §29]). $\Delta_{A,B}$ and $\nabla_{A,B}$ are moreover chain homotopy equivalences: we have a diagram

$$\begin{array}{ccccc} N(A) \otimes N(B) & \xrightarrow{\nabla_{A,B}} & N(A \otimes B) & & \\ & \searrow{=} & \downarrow{\scriptstyle \Delta_{A,B}} & \nearrow\!\!\!\!\nearrow \; \searrow{=} & \\ & & N(A) \otimes N(B) & \xrightarrow[\nabla_{A,B}]{} & N(A \otimes B), \end{array}$$

where the left triangle commutes on the nose, and the right up to chain homotopy [May92, Cor. 29.10]. Given a finite simplicial set K, the composite

$$N(\mathbb{Z}[K]) \to N(\mathbb{Z}[K] \otimes \mathbb{Z}[K]) \xrightarrow{\nabla_{K,K}} N(\mathbb{Z}[K]) \otimes N(\mathbb{Z}[K])$$

equips $N(\mathbb{Z}[K])$ with a (co)commutative counital coalgebra structure, where the first map is the image under the normalised chain complex functor of the diagonal map $\mathbb{Z}[K] \to \mathbb{Z}[K] \otimes \mathbb{Z}[K]$. The counit

$$N(\mathbb{Z}[K]) \to N(\mathbb{Z}[\Delta^0]) = \mathbb{Z}[0]$$

is induced by the unique map $K \to \Delta^0$ of simplicial sets. For $\Delta^\bullet$ the standard cosimplicial simplicial set, $N(\mathbb{Z}[\Delta^\bullet])$ is by naturality a cosimplicial counital cocommutative coalgebra.
Fix for the rest of this section a complicial exact category with weak equivalences $(\mathcal{E}, w, D, \eta)$. For a finite simplicial set K and $x \in \mathcal{E}$, we write $Kx := N\,\mathbb{Z}[K] \otimes x$.

Construction 3.1.4. Given a complicial exact category $(\mathcal{E}, \otimes)$, a presheaf F of abelian groups on $\mathcal{E}$, write $F^{\Delta^\bullet} : \mathcal{E}^{op} \to s\mathcal{A}b$ for the presheaf of simplicial abelian groups

$$F^{\Delta^\bullet}(x) := F(\Delta^\bullet x).$$

$F^{\Delta^\bullet}$ receives a natural transformation from F induced by the unique map of cosimplicial simplicial sets $\Delta^\bullet \to \Delta^0$.

Proposition 3.1.5. *The functor*

$$F^{\Delta^\bullet} : \mathcal{E}^{op} \to s\mathcal{A}b$$

sends Frobenius homotopies to simplicial homotopies.

Proof. Write $G := F^{\Delta^\bullet}$. We claim for each x that the composites

$$G(\Delta^1 x) \overset{G(d^1)}{\underset{G(d^0)}{\rightrightarrows}} G(x) \xrightarrow{\ p\ } G(\Delta^\bullet x),$$

are simplicially homotopic in $s\mathcal{A}b$, where p is induced upon taking diagonals by the map $\Delta^\bullet \to \Delta^0$. We have maps

$$\mathrm{Hom}_{s\mathcal{A}b}(\mathbb{Z}[\Delta^\bullet], \mathbb{Z}[\Delta^1]) \otimes G(\Delta^1 x) \to G(\Delta^\bullet x), \quad f \otimes \xi \mapsto G(f \otimes 1)(\xi),$$

and

$$\Delta^1 \to \mathrm{Hom}_{s\mathcal{A}b}(\mathbb{Z}[\Delta^\bullet], \mathbb{Z}[\Delta^1])$$

with the latter the image under Yoneda of $\mathrm{id}_{\mathbb{Z}[\Delta^1]} \in \mathrm{Hom}_{s\mathcal{A}b}(\mathbb{Z}[\Delta^1], \mathbb{Z}[\Delta^1])$. The composite

$$\mathbb{Z}[\Delta^1] \otimes G(\Delta^1 x) \to G(\Delta^\bullet x)$$

then gives the desired homotopy. The comultiplication $\nabla : \Delta^\bullet \to \Delta^\bullet \otimes \Delta^\bullet$ induces a map of simplicial abelian groups

$$G(\Delta^\bullet x) = F((\Delta^\bullet \otimes \Delta^\bullet)x) \xrightarrow{F(\nabla \otimes 1)} G(x),$$

and the composite

$$\Delta^1 \otimes G(\Delta^1 x) \to G(\Delta^\bullet x) \to G(x)$$

is a homotopy between $G(d^1)$ and $G(d^0)$. Given then a Frobenius homotopy witnessed by the diagram

$$\begin{array}{ccc} x & & \\ \downarrow{\scriptstyle d^1} & \searrow^{f} & \\ \Delta^1 x & \xrightarrow{H} & y. \\ \uparrow{\scriptstyle d^0} & \nearrow_{g} & \\ x & & \end{array}$$

we see that $G(f) = G(Hd^1) \sim G(Hd^0) = G(g)$. ☐

Remark 3.1.6. Recall that the free-forgetful adjunction

$$\mathrm{sSet} \underset{U}{\overset{\mathbb{Z}[-]}{\underset{\perp}{\rightleftarrows}}} s\mathcal{A}b,$$

tautologically refines to a Quillen adjunction with respect to the Kan-Quillen model structure on simplicial sets, and the projective model structure on $s\mathcal{A}b$ (for which the weak equivalences and fibrations are underlying). It follows that the composite

$$F^{\Delta^\bullet} : \mathcal{E}^{op} \to s\mathcal{A}b \xrightarrow{U} \mathrm{sSet}$$

sends Frobenius equivalences to weak equivalences of simplicial sets (in fact, to homotopy equivalences of Kan complexes).

Write $\mathrm{Ch}(\mathbb{Z})$ for the abelian category of (unbounded) complexes of abelian groups. By for instance [HA, Prop. 1.3.5.3], $\mathrm{Ch}(\mathbb{Z})$ admits a left proper cofibrantly generated combinatorial model structure with cofibrations the levelwise monomorphisms, and weak equivalences the quasi-isomorphisms, and we write $D(\mathbb{Z})$ for the underlying ∞-category, obtained for instance as the dg-nerve of the full subcategory $\mathrm{Ch}(\mathbb{Z})^\circ \subset \mathrm{Ch}(\mathbb{Z})$ of fibrant-cofibrant objects. $D(\mathbb{Z})$ is stable, presentable, and carries a natural t-structure with connectives (coconnectives) those complexes with vanishing homology in negative (positive) degrees; this t-structure is moreover accessible, right-complete, and such that the coconnective aisle $D_{\leq 0}(\mathbb{Z})$ is stable under small filtered colimits by [HA, Prop. 1.3.5.21]. We may identify $D_{\geq 0}(\mathbb{Z})$ with the animation of $\mathcal{A}b$, i.e. with the ∞-category $\mathrm{Fun}^{\Pi}(\mathrm{Free}^{fg}(\mathbb{Z})^{op}, \mathcal{S})$ of additive presheaves of spaces on the category of compact projective (i.e. finitely generated free) abelian groups, which coincides with the localisation of the category $s\mathcal{A}b$ of simplicial abelian groups at the weak equivalences [ČS24, Ex. 5.1.6(b)], and also with ∞-category of connective $H\mathbb{Z}$-modules $\mathrm{Mod}_{H\mathbb{Z},\geq 0}$ [Shi07, Th. 1.1].
For $\mathcal{E}$ a complicial exact category, write $\mathcal{E}_{\mathrm{Frob}}$ for the exact category with underlying category that of $\mathcal{E}$ equipped with the Frobenius exact structure of Remark A.3.6 (in which, informally, the ingressions $x \rightarrowtail y$ are those maps

which see objects of the form $C \otimes z$ for $z \in \mathcal{E}$ as injective, and dually for egressions). We have associated exact inclusions $(\mathcal{E}_{\mathrm{Frob}}, w_{\mathrm{Frob}}) \to (\mathcal{E}, w)$, localising to the Verdier quotient

$$\mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to \mathrm{L}_w(\mathcal{E})$$

by the discussion before Remark A.3.21. Now the presheaf $F^{\Delta^\bullet}$ descends to a presheaf on $\mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ taking values in $D_{\geq 0}(\mathbb{Z})$, which we also denote $F^{\Delta^\bullet}$. For $x \in \mathcal{E}$, write $J_{x,\mathrm{Frob}} \subset (\mathcal{E} \downarrow x)$ for the full subcategory spanned by the trivial Frobenius egressions over x. There is a functor $\iota_x : \mathbb{\Delta} \to J_{x,\mathrm{Frob}}$, sending $[n]$ to the codegeneracy map

$$\Delta^n x \overset{\sim}{\twoheadrightarrow} x,$$

and an arrow $[n] \xrightarrow{\theta} [m]$ in $\mathbb{\Delta}$ to

$$(\Delta^n x \overset{\sim}{\twoheadrightarrow} x) \xrightarrow{\theta_* \otimes 1} (\Delta^m x \overset{\sim}{\twoheadrightarrow}, x).$$

Proposition 3.1.7. ι_x^{op} *is cofinal; in particular for each* $x \in \mathcal{E}$*, there is a natural equivalence*

$$F^{\Delta^\bullet}(\gamma(x)) \simeq \operatorname*{colim}_{[n] \in \mathbb{\Delta}^{\mathrm{op}}} F(\Delta^n x) \xrightarrow{\simeq} \operatorname*{colim}_{J^{\mathrm{op}}_{x,\mathrm{Frob}}} F$$

in $D_{\geq 0}(\mathbb{Z})$.

Proof. It suffices by Quillen's Theorem A [HTT, Th. 4.1.3.1] to show that the slice categories

$$((y, p) \downarrow \iota_x^{\mathrm{op}}) \simeq (\iota_x \downarrow (y, p))^{\mathrm{op}}$$

have weakly contractible nerves for each trivial fibration $p : y \overset{\sim}{\twoheadrightarrow} x$ in $\mathcal{E}$. Objects of $(\iota_x \downarrow (y, p))^{\mathrm{op}}$ are pairs $([n] \in \mathbb{\Delta}, \gamma : \Delta^n x \to y)$, γ a map over x, with morphisms

$$([n], \gamma) \to ([m], \gamma')$$

arrows $\theta : [m] \to [n]$ in $\mathbb{\Delta}$ rendering the diagram

$$\begin{array}{ccccc} & & y & & \\ & \overset{\gamma'}{\nearrow} & \wr\big\uparrow p & \overset{\gamma}{\nwarrow} & \\ \Delta^m x & & \xrightarrow{\theta_* \otimes 1} & & \Delta^n x \\ & \underset{\sim}{\searrow} & \big\downarrow & \underset{\sim}{\swarrow} & \\ & & x & & \end{array} \tag{3.1}$$

commutative in $\mathcal{E}$. Since p is a trivial Frobenius egression, its kernel is contractible, hence injective-projective for the Frobenius exact structure on $\mathcal{E}$. The congression

$$\ker(p) \rightarrowtail y \twoheadrightarrow x$$

then splits, and p admits a section; the space $\mathrm{Sec}(p)$ of such sections is a contractible Kan complex by Lemma A.3.19.

Recall that the category of simplices of $\mathrm{Sec}(p)$ is the Grothendieck construction of the associated functor

$$\mathrm{Sec}(p) : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \subset \mathcal{C}\mathrm{at}, \quad [n] \mapsto \mathrm{Sec}_n(p) \subset \mathrm{Hom}_{\mathcal{E}}(\Delta^n x, y)$$

with objects pairs $([n], \xi \in \mathrm{Sec}_n(p))$, where an object of $\mathrm{Sec}_n(p)$ is a map $\Delta^n x \to y$ whose composition with p is the collapse map $\Delta^n x \xrightarrow{\sim} x$. Maps $([n], \gamma) \to ([m], \gamma')$ are maps $\theta : [m] \to [n]$ in $\mathbb{\Delta}$ such that the diagram (3.1) commutes, from which we see that there is a canonical identification of $\int_{\mathbb{\Delta}^{op}} \mathrm{Sec}(p)$ with $(\mathbb{\Delta} \downarrow (y,p))^{op}$. By Thomason's theorem ([Tho79, Th. 1.2], [HTT, Cor. 3.3.4.6]), it follows that

$$N((\mathbb{\Delta} \downarrow (y,p))^{op}) = N\left(\int_{\mathbb{\Delta}^{op}} \mathrm{Sec}(p)\right) \simeq \operatorname*{colim}_{[n]\in\mathbb{\Delta}^{op}} \mathrm{Sec}_n(p) \simeq \mathrm{Sec}(p) \simeq \Delta^0.$$

□

Remark 3.1.8. The same argument shows that, for $U : s\mathcal{A}b \to s\mathcal{S}et$ the forgetful functor, there is an equivalence of spaces

$$UF^{\Delta^\bullet}(\gamma(x)) \simeq \operatorname*{colim}_{J^{op}_{x,\mathrm{Frob}}} UF,$$

where we write U also for the restriction $\mathcal{A}b \to \mathcal{S}et$.

Corollary 3.1.9. $UF^{\Delta^\bullet}$ *is a left Kan extension of* UF *along (the opposite of) the localisation* $\gamma : \mathcal{E}_{\mathrm{Frob}} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$.

Proof. Recall (Lemma 3.1.2) that $(\mathcal{E}, \mathrm{Fib}, w_{\mathrm{Frob}})$ is an ∞-category of fibrant objects. Set $v_{\mathrm{Frob}} \subset w_{\mathrm{Frob}}$ the subcategory of trivial Frobenius egressions, and note that v_{Frob} is closed under composition and pullbacks in the sense of [Cis19, Def. 7.2.14]. Note moreover that the saturation $\overline{v_{\mathrm{Frob}}}$ of v_{Frob}, defined by the cartesian square

$$\begin{array}{ccc} \overline{v_{\mathrm{Frob}}} & \hookrightarrow & \mathcal{E}_{\mathrm{Frib}} \\ \downarrow & & \downarrow \gamma \\ L_{v_{\mathrm{Frob}}}(\mathcal{E}_{\mathrm{Frob}})^{\simeq} & \hookrightarrow & L_{v_{\mathrm{Frob}}}(\mathcal{E}_{\mathrm{Frob}}), \end{array}$$

contains w_{Frob}: any map $f : x \to y$ in $\mathcal{E}$ admits a factorisation

$$\begin{array}{ccc} x & \xrightarrow{\binom{1}{0}} & x \oplus Py \\ & \searrow^{f} & \downarrow\!\!\downarrow (\, f \;\; \pi_y \,) \\ & & y, \end{array}$$

where the horizontal map admits a retraction which is a trivial fibration, and the vertical map is a fibration. If f is a Frobenius equivalence, so are j and g, by 2-of-3. Accordingly, if $F : \mathcal{E}^{op} \to \mathcal{S}$ is a functor inverting maps in v_{Frob}, it also inverts maps in w_{Frob}. We then see from [Cis19, Th. 7.2.16, Cor. 7.2.18] that $J_{x,\mathrm{Frob}}$ is a right calculus of fractions at x in the sense of Cisinski, and accordingly by [Cis19, Cor. 7.2.9] and Proposition 3.1.7 the left Kan extension of $F : \mathcal{E}^{op} \to \mathcal{S}$ along the localisation γ satisfies

$$\gamma_! F(\gamma(x)) \simeq \operatorname*{colim}_{J^{op}_{x,\mathrm{Frob}}} F \simeq F^{\Delta^\bullet}(\gamma(x)).$$

□

Remark 3.1.10. (a) Recall [Cis19, §7.5.23] that given an ∞-category of fibrant objects $(\mathcal{C}, w)$ and a functor $\mathcal{C} \to \mathcal{D}$, then a right-derived functor of F is a pair $(\mathbf{R}F, \eta)$, for $\mathbf{R}F$ an object of $\mathrm{Fun}(L_w(\mathcal{C}), \mathcal{D})$ representing the functor

$$\mathrm{Fun}(L_w(\mathcal{C}), \mathcal{D}) \to \mathcal{S}, \quad G \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, \gamma^* G),$$

and $\eta : F \to \gamma^* \mathbf{R}F$ a natural transformation exhibiting this. In our case, we remark that since the opposite of a complicial exact category acquires a canonical complicial exact structure, $(\mathcal{E}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}_{\mathrm{in}}, w^{\mathrm{op}})$ is an ∞-category of fibrant objects. Since $F^{\Delta^\bullet} : \mathcal{E}^{\mathrm{op}} \to s\mathcal{A}b$ preserves weak equivalences, by [Cis19, Lem. 7.5.24] the induced functor $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to D_{\geq 0}(\mathbb{Z})$ is the right derived functor of the composition $L \circ F^{\Delta^\bullet}$ at the Frobenius equivalences, for $L : s\mathcal{A}b \to D_{\geq 0}(\mathbb{Z})$ the localisation.

(b) Recall [Cis19, §7.5.25] that given an ∞-category with weak equivalences and fibrations $(\mathcal{C}, w)$ and a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ inverting weak equivalences, then the right derived functor $\mathbf{R}F$ of F exists and identifies with the functor induced by the universal property of the localisation $\gamma : \mathcal{C} \to L_w(\mathcal{C})$. Moreover, for any $\mathcal{D} \xrightarrow{F'} \mathcal{D}'$, there is an equivalence

$$F' \circ \mathbf{R}F \simeq \mathbf{R}(F \circ F').$$

Accordingly, for functors $(\mathcal{C}_0, w_0) \xrightarrow{F_0} (\mathcal{C}_1, w_1) \xrightarrow{F_1} (\mathcal{C}_2, w_2)$ of ∞-categories with weak equivalences and fibrations, each preserving weak equivalences, we have an induced identification

$$\mathbf{R}(F_1) \circ \mathbf{R}(F_0) \simeq \mathbf{R}(F_1 \circ F_0).$$

In our case, we note the following: the forgetful functor $U : s\mathcal{A}b \to s\mathcal{S}et$ preserves weak equivalences, and admits a right derived functor $D_{\geq 0}(\mathbb{Z}) \to \mathcal{S}$ which identifies with the underlying space functor. Equipping $\mathcal{E}^{\mathrm{op}}$ with structure of an ∞-category of fibrant objects as above, with reference to the diagram

$$\begin{array}{ccccc} \mathcal{E}^{\mathrm{op}} & \xrightarrow{F^{\Delta^\bullet}} & s\mathcal{A}b & \xrightarrow{U} & s\mathcal{S}et \\ \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\ L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{\mathrm{op}} & \xrightarrow{\mathbf{R}(F^{\Delta^\bullet})} & D_{\geq 0}(\mathbb{Z}) & \xrightarrow{\mathbf{R}U} & \mathcal{S}, \end{array}$$

there is an identification $\mathbf{R}(U \circ F^{\Delta^\bullet}) \simeq \mathbf{R}U \circ \mathbf{R}(F^{\Delta^\bullet})$. In particular, the left Kan extension computing $\mathbf{R}(F^{\Delta^\bullet})$ is **absolute**, in the sense that it is preserved by any functor:

$$\mathbf{R}U\left(\operatorname*{colim}_{J^{\mathrm{op}}_{x,\mathrm{Frob}}} F\right) \simeq \left(\mathbf{R}U \circ \mathbf{R}(F^{\Delta^\bullet})\right)(\gamma(x)) \simeq \mathbf{R}(U \circ F^{\Delta^\bullet})(\gamma(x)) \simeq \operatorname*{colim}_{J^{\mathrm{op}}_{x,\mathrm{Frob}}} UF.$$

Remark 3.1.11. Recall from Appendix A.3 that any complicial exact category $\mathcal{E}$ is canonically enriched over simplicial abelian groups via

$$\mathrm{Map}_\Delta(x, y) := \mathrm{Hom}_{\mathcal{E}}(\Delta^\bullet x, y),$$

with homotopy coherent nerve a model for the localisation $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ at the Frobenius equivalences. Write $\mathcal{E}_\Delta$ for $s\mathcal{A}b$-enriched category thus obtained. Now $s\mathcal{A}b$ is canonically closed symmetric monoidal with internal mapping complexes $\underline{\mathrm{Hom}}(A, B) := \mathrm{Hom}_{s\mathcal{A}b}(\mathbb{Z}[\Delta^\bullet] \otimes A, B)$. The functor $F^{\Delta^\bullet}$ canonically enhances to a simplicially enriched functor

$$F^{\Delta^\bullet} : \mathcal{E}^{\mathrm{op}}_\Delta \to s\mathcal{A}b$$

via the map

$$\mathrm{Map}_\Delta(x, y) \to \underline{\mathrm{Hom}}(F^{\Delta^\bullet}(y), F^{\Delta^\bullet}(x))$$

given on n-simplices by

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{E}}(\Delta^n x, y) \to \mathrm{Hom}_{s\mathcal{A}b}(\mathbb{Z}[\Delta^n] \otimes F^{\Delta^\bullet}(y), F^{\Delta^\bullet}(x)), \\ &\qquad \alpha \mapsto (\mathbb{Z}[\Delta^n_k] \otimes F(\Delta^k y) \ni \sigma \otimes \xi \mapsto F((1 \otimes \alpha) \circ (1 \otimes \theta \otimes 1) \circ (\nabla^k_x)) \in F(\Delta^k x)), \end{aligned}$$

i.e. sending a map $\alpha : \Delta^n x \to y$ to the image under F of

$$\Delta^k x \xrightarrow{\nabla^k_x} (\Delta^k \otimes \Delta^k)x \xrightarrow{1\otimes\theta\otimes 1} (\Delta^k \otimes \Delta^n)x \xrightarrow{1\otimes\alpha} \Delta^k y.$$

Compatibility with composition and unit follow ultimately from the cosimplicial coalgebra structure on $\Delta^\bullet$. Taking homotopy-coherent nerves, we obtain a model for the induced presheaf on the localisation $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$; the same holds upon postcomposition with the simplicial functor $s\mathcal{A}b \to s\mathcal{S}et$.

3.2 The underlying hermitian ∞-category of a complicial exact form category

Given a complicial exact form category with weak equivalences $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$, we may apply the construction $F \mapsto F^{\Delta^\bullet}$ of the previous section to obtain a model $Q^{\Delta^\bullet} : \mathcal{E}^{\mathrm{op}} \to s\mathcal{A}b$ for the right derived functor of Q along the localisation $\mathcal{E}_{\mathrm{Frob}} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$. Note that for each $n \geq 0$ the C_2-Mackey functors

$$\mathrm{Hom}_{\mathcal{E}}(\Delta^n x, \mathbb{D}(\Delta^n x)) \xrightarrow{\tau_{\Delta^n x}} Q(\Delta^n x) \xrightarrow{\rho_{\Delta^n x}} \mathrm{Hom}_{\mathcal{E}}(\Delta^n x, \mathbb{D}(\Delta^n x))$$

of Definition 2.1.4(i) assemble into C_2-equivariant maps of simplicial abelian groups

$$\mathrm{Hom}_{\mathcal{E}}(\Delta^\bullet x, \mathbb{D}(\Delta^\bullet x)) \xrightarrow{\tau_{\Delta^\bullet x}} Q^{\Delta^\bullet}(x) \xrightarrow{\rho_{\Delta^\bullet x}} \mathrm{Hom}_{\mathcal{E}}(\Delta^\bullet x, \mathbb{D}(\Delta^\bullet x)),$$

where C_2 acts levelwise, so in particular

$$\mathrm{Hom}_{\mathcal{E}}(\Delta^\bullet x, \mathbb{D}(\Delta^\bullet x))^{C_2} = \left([n] \mapsto \mathrm{Hom}_{\mathcal{E}}(\Delta^n x, \mathbb{D}(\Delta^n x))^{C_2}\right),$$

and similarly for C_2-orbits.

Remark 3.2.1. Consider the subfunctor Q_{nd} of the restriction

$$w\mathcal{E}^{\mathrm{op}} \xrightarrow{Q} \mathcal{A}b \to \mathcal{S}et$$

with $Q_{\mathrm{nd}}(x) := \{\xi \in Q(x) \mid \rho_x(\xi) \in w\mathcal{E}\}$, i.e. given as the pullback (in $\mathcal{S}et$, and accordingly in $\mathcal{S}$)

$$\begin{array}{ccc} Q_{\mathrm{nd}}(x) & \hookrightarrow & Q(x) \\ \downarrow & & \downarrow \rho_x \\ \mathrm{Hom}_{w\mathcal{E}}(x, \mathbb{D}(x)) & \hookrightarrow & \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)). \end{array}$$

That this assignment is functorial in weak equivalences follows from the commutativity of

$$\begin{array}{ccc} Q(x) & \xrightarrow{\rho_x} & \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)) \\ \downarrow f^\bullet & & \downarrow \mathbb{D}(f)_* f^* \\ Q(y) & \xrightarrow{\rho_y} & \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(y)) \end{array}$$

for each map $f : y \to x$, and that $\mathbb{D}$ preserves equivalences. For $w = w_{\mathrm{Frob}}$, the proof of Proposition 3.1.7 and Remark 3.1.10 then imply that the map of spaces

$$Q^{\Delta^\bullet}_{\mathrm{nd}}(x) = Q_{\mathrm{nd}}(\Delta^\bullet x) \to \operatorname*{colim}_{J^{\mathrm{op}}_{x,\mathrm{Frob}}} Q_{\mathrm{nd}}$$

is an equivalence.

In the remainder of this section we prove that $Q^{\Delta^\bullet}$ gives a model for (the connective cover of) the 2-excisive (right) derived functor of Q on the stable ∞-category $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$. Recall that for $\mathcal{C}$ and $\mathcal{D}$ ∞-categories with finite colimits and limits respectively, a functor $\mathcal{Q}: \mathcal{C} \to \mathcal{D}$ is said to be 2-excisive if for each strongly cocartesian cube $C: N\mathcal{P}([2]) \to \mathcal{C}$, the cube $\mathcal{Q}(C)$ in $\mathcal{D}$ is a limit diagram. Recall from Appendix B.1 that for $\mathcal{C}$ a stable ∞-category, the inclusion

$$\mathrm{Fun}_*^{2\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p) \subset \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathcal{S}p)$$

admits a left adjoint P_2 (where Fun_* denotes on each side the full subcategory of reduced functors), computed as the sequential colimit

$$P_2\mathcal{R} \simeq \varinjlim(\mathcal{R} \to T_2\mathcal{R} \to T_2^2\mathcal{R} \to \dots),$$

where $T_2\mathcal{R}(x) := \Omega\,\mathrm{fib}(\mathcal{R}(\Omega x) \to B_{\mathcal{R}}(\Omega x, \Omega x))$. Here, $B_{\mathcal{R}}(x,y) := \mathrm{fib}(\mathcal{R}(x \oplus y) \to \mathcal{R}(x) \oplus \mathcal{R}(y))$ is the polarisation, and the map $\mathcal{R} \to T_2\mathcal{R}$ is pointwise the natural map from $\mathcal{R}(x)$ to the total fibre of the square

$$\begin{array}{ccc} 0 \simeq \mathcal{R}(0) & \longrightarrow & \mathcal{R}(\Omega x) \\ \downarrow & & \downarrow \\ 0 \simeq B_{\mathcal{R}}(0, \Omega x) & \longrightarrow & B_{\mathcal{R}}(\Omega x, \Omega x) \end{array}$$

arising from the exact sequence $\Omega x \to 0 \to x$. Now suppose given a functor $\mathcal{R}: \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p_{\geq 0}$ such that the map $\mathcal{R}(z) \to \mathcal{R}(y)$ exhibits the source as the total fibre (in $\mathcal{S}p_{\geq 0}$) of the square

$$\begin{array}{ccc} \mathcal{R}(y) & \longrightarrow & \mathcal{R}(x) \\ \downarrow & & \downarrow \\ B_{\mathcal{R}}(y, x) & \longrightarrow & B_{\mathcal{R}}(x, x) \end{array}$$

for each exact sequence $x \to y \to z$. Then we claim that $\mathcal{R}$ is 2-excisive: indeed, we may computed the 2-excisive approximation of the functor $\iota\mathcal{R}: \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p_{\geq 0} \subset \mathcal{S}p$ as $P_2\iota\mathcal{R}$, and since connective cover preserves total fibres, the functor $\tau_{\geq 0}P_2\iota\mathcal{R} \simeq \mathcal{R}$ is reduced and 2-excisive, since the natural map

$$\mathcal{R}(x) \to \mathrm{fibt}\left[\begin{array}{ccc} 0 & \longrightarrow & \mathcal{R}(\Omega x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B_{\mathcal{R}}(\Omega x, \Omega x) \end{array}\right]$$

is an equivalence by hypothesis.

Set

$$L_Q: \mathcal{E}^{\mathrm{op}} \to \mathcal{A}b, \quad x \mapsto \ker\left(Q(x) \xrightarrow{\rho_x} \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2}\right).$$

Lemma 3.2.2. L_Q *is left exact, i.e. sends each congression* $x \overset{i}{\rightarrowtail} y \overset{p}{\twoheadrightarrow} z$ *to a left-exact sequence of abelian groups*

$$0 \to L_Q(z) \xrightarrow{L_Q(p)} L_Q(y) \xrightarrow{L_Q(i)} L_Q(x).$$

Proof. Recall that Q and $[\Delta^* \operatorname{Hom}_{\mathcal{E}}(-, \mathbb{D}(-))]^{C_2}$ are quadratic left exact with the map $\rho : Q \Rightarrow [\Delta^* \operatorname{Hom}_{\mathcal{E}}(-, \mathbb{D}(-))]^{C_2}$ inducing an isomorphism on polarisations (Example 2.1.7). Given a congression as in the statement, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & L_Q(z) & \xrightarrow{L_Q(p)} & L_Q(y) & \xrightarrow{L_Q(i)} & L_Q(x) \\
 & & \downarrow{\scriptstyle \ker} & & \downarrow{\scriptstyle \ker} & & \downarrow{\scriptstyle \ker} \\
0 & \longrightarrow & Q(z) & \xrightarrow{p^\bullet} & Q(y) & \xrightarrow{\begin{pmatrix} i^\bullet \\ \mathbb{D}(i)_* \rho_y \end{pmatrix}} & Q(x) \oplus \operatorname{Hom}_{\mathcal{E}}(y, \mathbb{D}(x)) \\
 & & \downarrow{\scriptstyle \rho_z} & & \downarrow{\scriptstyle \rho_y} & & \downarrow{\scriptstyle \begin{pmatrix} \rho_x & 0 \\ 0 & 1 \end{pmatrix}} \\
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{E}}(z, \mathbb{D}(z))^{C_2} & \xrightarrow{p^* \mathbb{D}(p)_*} & \operatorname{Hom}_{\mathcal{E}}(y, \mathbb{D}(y))^{C_2} & \xrightarrow{\begin{pmatrix} i^* \mathbb{D}(i)_* \\ \mathbb{D}(i)_* \end{pmatrix}} & \operatorname{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2} \oplus \operatorname{Hom}_{\mathcal{E}}(y, \mathbb{D}(x))
\end{array}$$

exhibits L_Q as left exact. □

Set

$$B_{Q^{\Delta^\bullet}} : \mathcal{E}^{op} \times \mathcal{E}^{op} \to s\mathcal{A}b, \quad (x, y) \mapsto \operatorname{hofib}\left(Q^{\Delta^\bullet}(x \oplus y) \to Q^{\Delta^\bullet}(x) \oplus Q^{\Delta^\bullet}(y)\right),$$

where the homotopy fibre is taken in simplicial abelian groups with respect to the classical (projective) model structure. Since the inclusions $x \to x \oplus y \leftarrow y$ are split monomorphisms, the map $Q^{\Delta^\bullet}(x \oplus y) \to Q^{\Delta^\bullet}(x) \oplus Q^{\Delta^\bullet}(y)$ is a levelwise split surjection of simplicial abelian groups and hence a fibration [GJ09, Lem. III.2.8]. Accordingly, we have a natural zig-zag

$$B_{Q^{\Delta^\bullet}}(x, y) \simeq \operatorname{Hom}_{\mathcal{E}}(\Delta^\bullet x, \mathbb{D}(\Delta^\bullet y)) \simeq \operatorname{Map}_\Delta(x, \mathbb{D}(y)),$$

where the second equivalence is induced by the map of bisimplicial abelian groups given in bidegree (p, q) by

$$\operatorname{Hom}_{\mathcal{E}}(\Delta^p x, \mathbb{D}(y)) \to \operatorname{Hom}_{\mathcal{E}}(\Delta^p x, \mathbb{D}(\Delta^q y)) \tag{3.2}$$

induced by the codegeneracies $\Delta^q \to \Delta^0$. Since $\mathbb{D}$ preserves Frobenius equivalences (Definition A.3.4) and $\operatorname{Hom}_{\mathcal{E}}(\Delta^\bullet x, -)$ sends Frobenius equivalences to homotopy equivalences, (3.2) is a levelwise weak equivalence and hence an equivalence upon taking diagonals.

As before, $Q^{\Delta^\bullet}$ descends to a functor

$$Q^{\Delta^\bullet} : L_{\text{Frob}}(\mathcal{E}_{\text{Frob}})^{op} \to D_{\geq 0}(\mathbb{Z}),$$

and similarly $B_{Q^{\Delta^\bullet}}$ descends to a functor

$$B_{Q^{\Delta^\bullet}} : L_{\text{Frob}}(\mathcal{E}_{\text{Frob}})^{op} \times L_{\text{Frob}}(\mathcal{E}_{\text{Frob}})^{op} \to D_{\geq 0}(\mathbb{Z}).$$

Note that the functors $\mathcal{E}^{op} \to L_w(\mathcal{E})^{op}$ and $\mathcal{E}^{op} \times \mathcal{E}^{op} \to L_w(\mathcal{E})^{op} \times L_w(\mathcal{E})^{op}$ are Dwyer-Kan localisations by [Cis19, Prop. 7.1.7, 7.1.13], for any set of weak equivalences $w \subset \mathcal{E}$.

Lemma 3.2.3. *Suppose given an exact category $\mathcal{E}$, and a cosimplicial sequence*

$$x^\bullet \to y^\bullet \to z^\bullet$$

in $\mathcal{E}$ which is levelwise split exact. Then for any quadratic functor $Q : \mathcal{E}^{op} \to \mathcal{A}b$ with polarisation B, the map $Q(z^\bullet) \to Q(y^\bullet)$ exhibits $Q(z^\bullet)$ as the total fibre in $\mathbb{D}(\mathbb{Z})$ of the square

$$\begin{array}{ccc}
Q(y^\bullet) & \longrightarrow & Q(x^\bullet) \\
\downarrow & & \downarrow \\
B(y^\bullet, x^\bullet) & \longrightarrow & B(x^\bullet, x^\bullet).
\end{array} \tag{3.3}$$

Proof. By [Sch21, Lem. A.12], for each n the sequence of abelian groups

$$0 \to Q(z^n) \to Q(y^n) \to Q(x^n) \oplus B(y^n, x^n) \to B(x^n, x^n) \to 0$$

is exact, and so the map $Q(z^n) \to Q(y^n)$ exhibits $Q(z^n)$ as the total kernel of the square

$$\begin{array}{ccc} Q(y^n) & \longrightarrow & Q(x^n) \\ \downarrow & & \downarrow \\ B(y^n, x^n) & \longrightarrow & B(x^n, x^n). \end{array}$$

Surjectivity of $Q(x^n) \oplus B(y^n, x^n) \to B(x^n, x^n)$ implies in fact that $HQ(z^n) \to HQ(y^n)$ exhibits the former as the total fibre in $D(\mathbb{Z})$ of the corresponding square of Eilenberg-Mac Lane spectra. Since fibre-cofibre sequences of spectra commute with realisation, the map $Q(z^\bullet) \to Q(y^\bullet)$ thus exhibits the former as the total fibre in $D(\mathbb{Z})$ of (3.3). □

Proposition 3.2.4. *For each Frobenius exact sequence* $x \rightarrowtail y \twoheadrightarrow z$ *in* $\mathcal{E}$*, the map* $Q^{\Delta^\bullet}(z) \to Q^{\Delta^\bullet}(y)$ *exhibits the source as the total fibre of the square*

$$\begin{array}{ccc} Q^{\Delta^\bullet}(y) & \longrightarrow & Q^{\Delta^\bullet}(x) \\ \downarrow & & \downarrow \\ B_{Q^{\Delta^\bullet}}(y, x) & \longrightarrow & B_{Q^{\Delta^\bullet}}(x, x). \end{array}$$

Proof. For each exact sequence $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{E}$, the following sequence is exact by hypothesis

$$0 \to Q(z) \to Q(y) \to Q(x) \oplus B_Q(y, x) \to B_Q(x, x),$$

and so the sequence

$$0 \to Q(y)/Q(z) \to Q(x) \oplus B_Q(y, x) \to B_Q(x, x)$$

is exact. Now for each $x \in \mathcal{E}$ and $n \geq 0$, write $\tilde{x}^n := \ker(\Delta^n x \twoheadrightarrow x)$ (note that $\tilde{x}^0 = 0$). Since $\Delta^n x \twoheadrightarrow x$ is a Frobenius egression, $\tilde{x}^n$ is Frobenius contractible, and the cosimplicial sequence $\tilde{x}^\bullet \to \Delta^\bullet x \to x$ is levelwise split, so that we have a (strict) pullback square of simplicial abelian groups

$$\begin{array}{ccc} \frac{Q^{\Delta^\bullet}(x)}{Q(x)} & \twoheadrightarrow & Q(\tilde{x}^\bullet), \\ \downarrow & & \downarrow \\ B_Q(\Delta^\bullet x, \tilde{x}^\bullet) & \twoheadrightarrow & B_Q(\tilde{x}^\bullet, \tilde{x}^\bullet) \end{array}$$

with levelwise surjective horizontal maps. Now $B_Q(\Delta^\bullet x, \tilde{x}^\bullet)$ is contractible by virtue of the fibre sequence

$$B_Q(\Delta^\bullet x, x) \to B_Q(\Delta^\bullet x, \Delta^\bullet x) \to B_Q(\Delta^\bullet x, \tilde{x}^\bullet),$$

in which the first map is an equivalence (and since $\pi_0 B_Q(\Delta^\bullet x, \tilde{x}^\bullet) = 0$, this is a fibre sequence in $\mathcal{S}p$), so that we have a fibre sequence

$$\frac{Q^{\Delta^\bullet}(x)}{Q(x)} \to Q(\tilde{x}^\bullet) \to B_Q(\tilde{x}^\bullet, \tilde{x}^\bullet),$$

which since $\pi_0 B_Q(\tilde{x}^\bullet, \tilde{x}^\bullet) = 0$ is a fibre sequence of spectra. Now considered as a fibre sequence in $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{S}p)$, the second and third term exhibit the desired behaviour on total fibres, since Q and $\Delta^* B_Q$ exhibit the correct total fibre behaviour on levelwise split-exact cosimplicial sequences, and for a Frobenius exact sequence $x \rightarrowtail y \twoheadrightarrow z$, the sequence $\tilde{x}^\bullet \rightarrowtail \tilde{y}^\bullet \twoheadrightarrow \tilde{z}^\bullet$ is levelwise split exact, being pointwise Frobenius contractible. Accordingly, $\frac{Q^{\Delta^\bullet}(x)}{Q(x)}$ exhibits the correct behaviour on total fibres.

Now consider the map of squares

$$\begin{array}{ccc} Q(y) \longrightarrow Q(x) & & Q^{\Delta^\bullet}(y) \longrightarrow Q^{\Delta^\bullet}(x) & & \frac{Q^{\Delta^\bullet}(y)}{Q(y)} \longrightarrow \frac{Q^{\Delta^\bullet}(x)}{Q(x)} \\ \downarrow \qquad\qquad \downarrow & \Rightarrow & \downarrow \qquad\qquad \downarrow & \Rightarrow & \downarrow \qquad\qquad \downarrow \\ B_Q(y,x) \longrightarrow B_Q(x,x) & & B_{Q^{\Delta^\bullet}}(y,x) \longrightarrow B_{Q^{\Delta^\bullet}}(x,x) & & \frac{B_{Q^{\Delta^\bullet}}(y,x)}{B_Q(y,x)} \longrightarrow \frac{B_{Q^{\Delta^\bullet}}(x,x)}{B_Q(x,x)} \end{array}$$

which is levelwise an exact sequence of simplicial abelian groups in each corner, and accordingly a fibre sequence of spectra in each corner, since the rightmost terms each have trivial π_0 (since their vertex abelian group is zero). This sequence receives a map from the fibre sequence of squares

$$\begin{array}{ccc} Q(z) \longrightarrow 0 & & Q^{\Delta^\bullet}(z) \longrightarrow 0 & & \frac{Q^{\Delta^\bullet}(z)}{Q(z)} \longrightarrow 0 \\ \downarrow \qquad \downarrow & \Rightarrow & \downarrow \qquad \downarrow & \Rightarrow & \downarrow \qquad \downarrow \\ 0 \longrightarrow 0 & & 0 \longrightarrow 0 & & 0 \longrightarrow 0, \end{array}$$

and on total fibres (taken in spectra), there is a map of fibre sequences

$$\begin{array}{ccccc} Q(z) & \longrightarrow & Q^{\Delta^\bullet}(z) & \longrightarrow & \frac{Q^{\Delta^\bullet}(z)}{Q(z)} \\ \downarrow & & \downarrow & & \| \\ G & \longrightarrow & F & \longrightarrow & \frac{Q^{\Delta^\bullet}(z)}{Q(z)}. \end{array}$$

Now $Q(z) \to G$ is a π_0-equivalence since Q is quadratic left-exact, and $\pi_0 \frac{Q^{\Delta^\bullet}(z)}{Q(z)} = 0$, so this remains a map of fibre sequences in $\mathcal{S}p$ after taking connective covers. We then see that $Q^{\Delta^\bullet}(z) \to \tau_{\geq 0}F$ is an equivalence, i.e. the map $Q^{\Delta^\bullet}(z) \to Q^{\Delta^\bullet}(y)$ exhibits $Q^{\Delta^\bullet}$ as the total homotopy fibre in simplicial abelian groups of the square

$$\begin{array}{ccc} Q^{\Delta^\bullet}(y) & \longrightarrow & Q^{\Delta^\bullet}(x) \\ \downarrow & & \downarrow \\ B_{Q^{\Delta^\bullet}}(y,x) & \longrightarrow & B_{Q^{\Delta^\bullet}}(x,x). \end{array}$$

□

Corollary 3.2.5. $Q^{\Delta^\bullet} : L_{\mathrm{Frob}}(\mathcal{E})^{\mathrm{op}} \to D_{\geq 0}(\mathbb{Z})$ *is reduced and 2-excisive.*

Postcomposing with the inclusion $D_{\geq 0}(\mathbb{Z}) \hookrightarrow D(\mathbb{Z})$, we obtain by abuse of notation a functor

$$Q^{\Delta^\bullet} : L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{\mathrm{op}} \to D(\mathbb{Z}),$$

which is reduced but in general fails spectacularly to be 2-excisive[2]. Write $\mathrm{Fun}^q := \mathrm{Fun}_*^{2-\mathrm{exc}}$, and $\mathbf{R}_{\mathrm{Frob}}Q := P_2 Q^{\Delta^\bullet}$. We have adjunctions

$$\mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, D_{\geq 0}(\mathbb{Z})) \underset{(\tau_{\geq 0})_*}{\overset{j_*}{\rightleftarrows}} \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, D(\mathbb{Z})) \underset{i}{\overset{P_2}{\rightleftarrows}} \mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, D(\mathbb{Z}))$$

[2]In the case $\mathcal{E} = \mathrm{Ch}_b(\mathcal{D})$ for some exact category $\mathcal{D}$, and $Q = R_{\mathrm{ch}}$ is extended as in 2.2 from a quadratic functor R on $\mathcal{D}$, one inspects from the definition of R_{ch} that R_{ch} vanishes on strictly connective complexes. If $R_{\mathrm{ch}} : K_b(\mathcal{D})^{\mathrm{op}} \to \mathcal{S}p$ were 2-excisive, it would be zero: indeed, this follows from the cofibre sequence $\Sigma B_{\mathfrak{P}}(\Sigma x, \Sigma x) \to \Sigma \mathfrak{P}(\Sigma x) \to \mathfrak{P}(x)$ for any 2-excisive $\mathfrak{P}$ ([CDH+I, Cons. 1.1.26]), the observation that $\delta^* B_{\mathfrak{P}}$ is a summand of $\mathfrak{P}$ and hence also vanishes on $K_b(\mathcal{D})_{\geq 1}$, and boundedness of the weight structure on $K_b(\mathcal{D})$.

where we write $j : D_{\geq 0}(\mathbb{Z}) \hookrightarrow D(\mathbb{Z})$ for the inclusion, left adjoint to $\tau_{\geq 0}$. Pointwise connective cover sends 2-excisive $D(\mathbb{Z})$-valued functors to 2-excisive $D_{\geq 0}(\mathbb{Z})$-valued functors, and so this restricts to an adjunction

$$\mathrm{Fun}^{q}(\mathcal{C}^{op}, D_{\geq 0}(\mathbb{Z})) \underset{(\tau_{\geq 0})_* \circ i}{\overset{P_2 j_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Fun}^{q}(\mathcal{C}^{op}, D(\mathbb{Z})).$$

The following is also noted in a more general form in [HSV21, Prop. 6.14].

Proposition 3.2.6. *For any stable ∞-category $\mathcal{C}$, the above is an adjoint equivalence. In particular, for any complicial exact form category with weak equivalences $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$, the natural map $Q^{\Delta^\bullet} \to \mathbf{R}_{\mathrm{Frob}}Q$ induces an equivalence $Q^{\Delta^\bullet} \xrightarrow{\simeq} \tau_{\geq 0}\mathbf{R}_{\mathrm{Frob}}Q$ on connective covers of functors $\mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{op} \to D_{\geq 0}(\mathbb{Z})$.*

Proof. Suppose given some $\mathfrak{P} : \mathcal{C}^{op} \to D_{\geq 0}(\mathbb{Z})$ 2-excisive, the unit

$$\mathfrak{P} \to \tau_{\geq 0,*} i P_2 j_* \mathfrak{P}$$

is given pointwise by

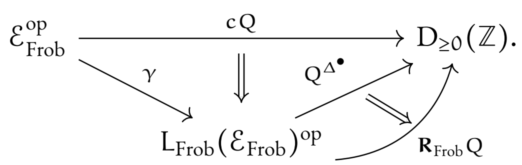
$$\mathfrak{P}(x) \to \tau_{\geq 0} P_2 j \mathfrak{P}(x) \simeq \tau_{\geq 0} \varinjlim_n T_2^n j \mathfrak{P}(x),$$

and since $\tau_{\geq 0}$ commutes with filtered colimits, to show this is an equivalence it suffices to show that $\tau_{\geq 0} T_2^{D(\mathbb{Z})} j \simeq T_2^{D_{\geq 0}(\mathbb{Z})} \tau_{\geq 0} j \simeq T_2^{D_{\geq 0}(\mathbb{Z})}$, where the superscript indicates where the fibre is taken. But

$$T_2 \mathfrak{P}(x) = \Omega\, \mathrm{fib}(\mathfrak{P}(\Omega x) \to B_{\mathfrak{P}}(\Omega x, \Omega x)),$$

and $\tau_{\geq 0}$, as a right adjoint, preserves fibre sequences. The left adjoint $P_2 j_*$ is consequently fully faithful, and so it suffices to show that $\tau_{\geq 0_*}$ is conservative; this follows from [CDH⁺I, Lem. 1.1.25]. □

Remark 3.2.7. Let $(\mathcal{E}, Q, w)$ be a complicial exact form category with weak equivalences. We have a canonical natural transformation $Q \Rightarrow Q^{\Delta^\bullet} \Rightarrow \mathbf{R}_{\mathrm{Frob}}Q$, with the first map given by the inclusion of 0-simplices, and the second the unit of the adjunction $P_2 \dashv i$. Accordingly, there are 2-cells in $\mathcal{C}at_\infty$



Writing $Q_{nd} \subset Q$ and $(\mathbf{R}_{\mathrm{Frob}}Q)_{nd} \subset \Omega^\infty \mathbf{R}_{\mathrm{Frob}}Q$ for the subfunctors of nondegenerate forms, this restricts to a natural transformation of functors $w\mathcal{E}^{op} \to \mathcal{S}$

$$Q_{nd} \Rightarrow \gamma^*(\mathbf{R}_{\mathrm{Frob}}Q)_{nd}.$$

Write $B_{\mathbf{R}_{\mathrm{Frob}}Q}$ for the polarisation of $\mathbf{R}_{\mathrm{Frob}}Q$, given objectwise by

$$B_{\mathbf{R}_{\mathrm{Frob}}Q}(x, y) := \mathrm{fib}\left(\mathbf{R}_{\mathrm{Frob}}Q(x \oplus y) \to \mathbf{R}_{\mathrm{Frob}}Q(x) \oplus \mathbf{R}_{\mathrm{Frob}}Q(y)\right),$$

and $\mathbb{D} : \mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{op} \to \mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ for the functor induced by $\mathbb{D} : \mathcal{E}^{op} \to \mathcal{E}$ under localisation (recall that $\mathbb{D}$ preserves Frobenius equivalences). Recall that mapping spaces in any stable ∞-category $\mathcal{C}$ canonically enhance to mapping spectra, via [HA, Cor. 1.4.2.23]: the functor $\Omega^\infty : \mathcal{S}p \to \mathcal{S}$ induces an equivalence of ∞-categories

$$\Omega^\infty_* : \mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{S}p) \xrightarrow{\simeq} \mathrm{Fun}^{lex}(\mathcal{C}, \mathcal{S}),$$

where we write $\mathrm{Fun}^{\mathrm{ex}}$ for the subcategory of exact functors, and $\mathrm{Fun}^{\mathrm{lex}}$ for the subcategory of left-exact (finite limit preserving) functors. For $x \in \mathcal{C}$, write $\mathrm{map}_{\mathcal{C}}(x,-)$ for the functor corresponding under this equivalence to $\mathrm{Map}_{\mathcal{C}}(x,-)$.

Proposition 3.2.8. *There are equivalences of spectra*

$$B_{\mathbf{R}_{\mathrm{Frob}}Q}(x,y) \simeq \mathrm{map}_{L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})}(x, \mathbb{D}(y)),$$

natural in x *and* y.

Proof. Recall that the 2-excisive approximation $P_2 Q^{\Delta^{\bullet}}$ is computed pointwise in $D(\mathbb{Z})$ as the sequential colimit

$$P_2 Q^{\Delta^{\bullet}}(x) \simeq \mathrm{colim}(Q^{\Delta^{\bullet}}(x) \to T_2 Q^{\Delta^{\bullet}}(x) \to T_2^2 Q^{\Delta^{\bullet}}(x) \to \dots),$$

where $T_2 Q^{\Delta^{\bullet}}(x) = \Omega\,\mathrm{fib}\big(Q^{\Delta^{\bullet}}(\Omega x) \to B_{Q^{\Delta^{\bullet}}}(\Omega x, \Omega x)\big)$. We compute

$$B_{T_2 Q^{\Delta^{\bullet}}}(x,y) \simeq \Omega\,\mathrm{fibt}\begin{pmatrix} Q^{\Delta^{\bullet}}(\Omega x \oplus \Omega y) & \longrightarrow & B_{Q^{\Delta^{\bullet}}}(\Omega x \oplus \Omega y, \Omega x \oplus \Omega y) \\ \downarrow & & \downarrow \\ Q^{\Delta^{\bullet}}(\Omega x) \oplus Q^{\Delta^{\bullet}}(\Omega y) & \longrightarrow & B_{Q^{\Delta^{\bullet}}}(\Omega x, \Omega x) \oplus B_{Q^{\Delta^{\bullet}}}(\Omega y, \Omega y) \end{pmatrix}$$

$$\begin{aligned} &\simeq \Omega\,\mathrm{fib}\Big(B_{Q^{\Delta^{\bullet}}}(\Omega x, \Omega y) \xrightarrow{\Delta} B_{Q^{\Delta^{\bullet}}}(\Omega x, \Omega y) \oplus B_{Q^{\Delta^{\bullet}}}(\Omega y, \Omega x)\Big) \\ &\simeq \Omega^2 B_{Q^{\Delta^{\bullet}}}(\Omega x, \Omega y) \simeq \Omega^2\,\mathrm{Map}_{\Delta}(\Omega x, \Sigma\,\mathbb{D}(y)), \end{aligned}$$

using that $\mathbb{D}(\Omega y)) \simeq \Sigma\,\mathbb{D}(y))$. Inductively, we have

$$B_{T_2^n Q^{\Delta^{\bullet}}}(x,y) \simeq \Omega^{2n} B_{Q^{\Delta^{\bullet}}}(\Omega^n x, \Sigma^n\,\mathbb{D}(y)) \simeq \Omega^{2n} B_{Q^{\Delta^{\bullet}}}(\Omega^{2n} x, \mathbb{D}(y)),$$

and in the colimit,

$$B_{\mathbf{R}_{\mathrm{Frob}}Q}(x,y) \simeq P_1(\mathrm{Map}(x,-))\,\mathbb{D}(y)),$$

for P_1 the 1-excisive approximation of Appendix B.3. But this is precisely the mapping spectrum $\mathrm{map}(x, \mathbb{D}(y))$ in $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$: for $x \in \mathcal{E}$ and an exact functor $F : L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to \mathcal{S}\mathrm{p}$ we have a string of equivalences

$$\begin{aligned} &\mathrm{Nat}_{\mathrm{Fun}^{\mathrm{ex}}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}),\mathcal{S}\mathrm{p})}(\mathrm{map}(x,-), F) \\ \simeq &\mathrm{Nat}_{\mathrm{Fun}^{\mathrm{lex}}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}),\mathcal{S}\mathrm{p}_{\geq 0})}(\mathrm{Map}(x,-), \tau_{\geq 0}F) \\ \simeq &\mathrm{Nat}_{\mathrm{Fun}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}),\mathcal{S}\mathrm{p})}(\mathrm{Map}(x,-), F) \\ \simeq &\mathrm{Nat}_{\mathrm{Fun}^{\mathrm{ex}}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}),\mathcal{S}\mathrm{p})}(P_1\,\mathrm{Map}(x,-), F). \end{aligned}$$

□

Recall [CDH+II, §1.1] that the Verdier quotient $\pi : L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to L_w(\mathcal{E})$ is an exact functor of stable ∞-categories, sitting in a fibre-cofibre sequence

$$L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^w \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \xrightarrow{\pi} L_w(\mathcal{E})$$

in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$. Here, $\mathcal{E}^w \subset \mathcal{E}$ is the full subcategory of w-acyclic objects and $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^w \subset L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ the essential image of the subcategory of acyclics under the localisation $\mathcal{E}_{\mathrm{Frob}} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$. This is closed under retracts (see the discussion below Remark A.3.10), and any retract diagram in $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^w$ can be lifted to $\mathcal{E}^w$. By [CDH⁺I, Lem. 1.4.1(iii)], left Kan extension along an exact functor preserves preserves quadraticity (in the stable setting), and so writing $\mathbf{R}Q := \pi_! \mathbf{R}_{\mathrm{Frob}} Q$, the pair $(L_w(\mathcal{E}), \mathbf{R}Q)$ is again the data of a hermitian ∞-category.

Theorem 3.2.9. *For* $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ *a complicial exact form category with weak equivalences, the pair* $(L_w(\mathcal{E}), \mathbf{R}Q)$ *is the data of a Poincaré* ∞*-category. Moreover, for each* $x \in \mathcal{E}$*,* $\mathbf{R}Q$ *satisfies*

$$\tau_{\geq 0} \mathbf{R}Q(\gamma(x)) \simeq \operatorname*{colim}_{J_x^{\mathrm{op}}} Q,$$

for $J_x \subset (\mathcal{E} \downarrow x)$ *the full subcategory spanned by trivial egressions over* x*.*

Proof. Suppose firstly that w is the class of Frobenius equivalences. In this case, $\mathbf{R}Q = \mathbf{R}_{\mathrm{Frob}} Q$ is reduced and 2-excisive by Corollary 3.2.5, with nondegenerate polarisation by Proposition 3.2.8; it remains to show that the induced duality $\mathbb{D} : L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{\mathrm{op}} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ is an equivalence. Since the unit $\mathrm{id}_{\mathcal{E}} \Rightarrow \mathbb{D}\mathbb{D}^{\mathrm{op}}$ is a Frobenius equivalence by assumption, the induced functor $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{\mathrm{op}} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$ is an equivalence. For the general case, we note that the polarisation $B_{\mathbf{R}Q} \simeq (\pi_! \times \pi_!) B_{\mathbf{R}_{\mathrm{Frob}} Q}$ is nondegenerate by the formula for mapping spaces in a Verdier quotient [NS18, Th. I.3.3], and that the induced duality $\mathbb{D} : L_w(\mathcal{E})^{\mathrm{op}} \to L_w(\mathcal{E})$ is an equivalence since again the unit and counit are natural weak equivalences. The last claim then follows from Proposition A.3.22 (or Proposition 3.1.7 if $w = w_{\mathrm{Frob}}$), since the truncation $\tau_{\geq 0}$ commutes with the filtered colimit appearing in the formula for left Kan extension along $L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}) \to L_w(\mathcal{E})$. □

Remark 3.2.10. If the double dual identification η on $\mathcal{E}$ is a natural Frobenius equivalence[3], the claim for general w follows immediately from that for w_{Frob}: by [CDH⁺II, Ex. 1.1.7], for $(\mathcal{C}, \mathfrak{Q})$ a Poincaré category and $p : \mathcal{C} \to \mathcal{D}$ a Verdier projection, the pair $(\mathcal{D}, p_! \mathfrak{Q})$ is Poincaré if and only if $\ker(p)$ is closed under the duality $\mathbb{D}_{\mathfrak{Q}}$. Note for this that $\mathcal{E}^w \subset \mathcal{E}$ is closed under the (exact) duality functor.

Remark 3.2.11. The functor $\mathbb{D} : L_w(\mathcal{E})^{\mathrm{op}} \to L_w(\mathcal{E})$ participates in an adjunction exhibiting $(L_w(\mathcal{E}), \mathbb{D})$ as a stable ∞-category with perfect duality: equipping $\mathcal{E}$ with the canonical structure of an ∞-category of fibrant objects, so that $\mathcal{E}^{\mathrm{op}}$ inherits the structure of an ∞-category of cofibrant objects, we then obtain from [Cis19, Th. 7.5.39] that the adjunction $\mathbb{D}^{\mathrm{op}} \dashv \mathbb{D}$ localises to an adjunction

$$L_w(\mathcal{E})^{\mathrm{op}} \underset{\mathbf{R}\mathbb{D}}{\overset{\mathbf{L}\mathbb{D}}{\underset{\perp}{\rightleftarrows}}} L_w(\mathcal{E}),$$

and $\mathbf{R}\mathbb{D}$ identifies with $\mathbf{L}\mathbb{D}$ since these are each induced by the universal property of the localisation (and $L_w(\mathcal{E}^{\mathrm{op}}) \simeq L_w(\mathcal{E})^{\mathrm{op}}$).

3.3 Functoriality

Write $\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$ for the category of small exact categories and exact functors, and $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}$ for the category of small complicial exact categories with weak equivalences and exact functors between them. Recall from Remark

[3] This is the case for instance for the exact form category $(\mathrm{Ch_b}(\mathcal{E}), Q_{\mathrm{ch}}, \mathbf{qis}, \mathbb{D}, \eta)$ extended from $(\mathcal{E}, Q, \mathbb{D}, \eta)$

2.1.9 the definition of $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}$ as the subcategory of the Grothendieck construction on the functor

$$\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}^{\mathrm{op}} \to \mathrm{CAT}_1, \quad (\mathcal{E}, w) \mapsto \mathrm{Fun}^{\mathrm{q}}(\mathcal{E})$$

spanned by those tuples $(\mathcal{E}, w, Q)$ for which Q is quadratic left exact, and the polarisation $B_Q(-,-) \cong \mathrm{Hom}(-, \mathbb{D}(-))$ induces a duality $(\mathbb{D}, \eta)$ on $\mathcal{E}$, with maps the nonsingular exact form functors. There are functors

$$\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \to \mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \xrightarrow{L_{(-)}} \mathcal{C}\mathrm{at}_\infty,$$
$$\mathcal{E} \mapsto (\mathrm{Ch}_b(\mathcal{E}), \mathbf{qis}), (\mathcal{E}, w) \mapsto L_w(\mathcal{E})$$

where the latter is the composite $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$, which by Proposition A.3.13 factors through the forgetful functor $\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty \subset \mathcal{C}\mathrm{at}_\infty$. The functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$ is the composite

$$\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{N} \mathrm{s}\mathcal{S}\mathrm{et}^+ \to \mathcal{C}\mathrm{at}_\infty$$

for $\mathrm{s}\mathcal{S}\mathrm{et}^+$ the category of marked simplicial sets with marked model structure [HTT, §3], with underlying ∞-category $\mathcal{C}\mathrm{at}_\infty$, and $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ the ordinary category of pairs $(\mathcal{C}, W)$, for $\mathcal{C}$ a category and $W \subset \mathcal{C}$ a wide subcategory, with maps $(\mathcal{C}, W) \to (\mathcal{D}, V)$ functors $f : \mathcal{C} \to \mathcal{D}$ such that $f(W) \subset V$. The functor N sends $(\mathcal{C}, W)$ to $(N(\mathcal{C}), N(W))$, and the second is the canonical localisation, which can be modelled for instance as fibrant replacement in $\mathrm{s}\mathcal{S}\mathrm{et}^+$ [Hin16, §1.1].

Proposition 3.3.1. *There is a functor* $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}^{\mathrm{p}}_\infty$*, sending a complicial exact form category with weak equivalences* $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ *to the Poincaré ∞-category* $(L_w(\mathcal{E}), \mathbf{R}Q)$ *defined above.*

Proof. We have functors

$$\mathrm{Fun}^{\mathrm{qlex}} : (\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at})^{\mathrm{op}} \to \mathrm{CAT}_\infty, \quad (\mathcal{E}, w) \mapsto \mathrm{Fun}^{\mathrm{qlex}}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}\mathrm{b}),$$
$$\mathrm{Fun}^{\mathrm{q}} : (\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty)^{\mathrm{op}} \to \mathrm{CAT}_\infty, \quad \mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{q}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{p}),$$

where $\mathrm{Fun}^{\mathrm{qlex}}$ and $\mathrm{Fun}^{\mathrm{q}}$ denote the full subcategories of the functor categories $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}\mathrm{b})$ and $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, D(\mathbb{Z}))$ spanned by quadratic left-exact resp. quadratic functors. The functor $\mathrm{Fun}^{\mathrm{q}}$ unstraightens to the ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{h}}_\infty = \int_{\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty} \mathrm{Fun}^{\mathrm{q}}$ of hermitian categories, and we write $\mathcal{C}\mathrm{at}^{\mathrm{h},w}_{1,\mathrm{comp}}$ for the unstraightening $\int_{\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}} \mathrm{Fun}^{\mathrm{qlex}}$. The localisation functor $L_{(-)} : \mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$ induces a map of right fibrations

$$\int_{\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}} L^*_{(-)} \mathrm{Fun}^{\mathrm{q}} \to \mathcal{C}\mathrm{at}^{\mathrm{h}}_\infty,$$

and composing with the functor $\mathcal{C}\mathrm{at}^{\mathrm{h},w}_{1,\mathrm{comp}} \to \int_{\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{E}\mathrm{x}\mathcal{C}\mathrm{at}} L^*_{(-)} \mathrm{Fun}^{\mathrm{q}}$ induced on unstraightenings by the composite

$$\mathrm{Fun}^{\mathrm{qlex}}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}\mathrm{b}) \xrightarrow{\gamma_!(-)^{\Delta^\bullet}} \mathrm{Fun}^{\mathrm{q}}(L_w(\mathcal{E})^{\mathrm{op}}, D_{\geq 0}(\mathbb{Z})) \xrightarrow{P_2} \mathrm{Fun}^{\mathrm{q}}(L_w(\mathcal{E})^{\mathrm{op}}, D(\mathbb{Z}))$$

for $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ the Dwyer-Kan localisation, we obtain a functor $\mathcal{C}\mathrm{at}^{\mathrm{h},w}_{1,\mathrm{comp}} \to \mathcal{C}\mathrm{at}^{\mathrm{h}}_\infty$, which on objects sends $(\mathcal{E}, Q) \mapsto (L_w(\mathcal{E}), P_2\gamma_! Q^{\Delta^\bullet} =: \mathbf{R}Q)$. We have a faithful embedding $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}^{\mathrm{h},w}_{1,\mathrm{comp}}$. Given a tuple $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ in $\mathrm{w}\mathcal{C}\mathrm{omp}\mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}$, the pair $(L_w(\mathcal{E}), \mathbf{R}Q)$ is a Poincaré ∞-category by 3.2.9, and given a nonsingular form functor

$$(f, \varphi_q, \varphi) : (\mathcal{E}, Q, w, \mathbb{D}, \eta) \to (\mathcal{E}', Q', w', \mathbb{D}', \eta'),$$

the induced functor $F : L_w(\mathcal{E}) \to L_{w'}(\mathcal{E}')$ on localisations is then duality preserving since φ is a natural weak equivalence. □

3.4 Poincaré structures on derived ∞-categories

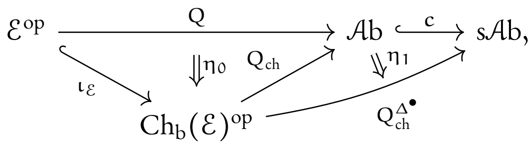
We now specialise the above to the case of form exact categories of chain complexes. Analogously to the story for K-theory, the passage from an exact category to a stable ∞-category goes via the (bounded) derived ∞-category $D_b(\mathcal{E})$ (or stable envelope), as recalled in Appendix A.1. Fix an exact form category $(\mathcal{E}, Q, \mathbb{D}, \eta)$. Recall from §2.2 that the exact embedding $\mathcal{E} \hookrightarrow \mathrm{Ch}_b(\mathcal{E})$, $x \mapsto x[0]$, enhances to an exact form functor of exact form categories with weak equivalences

$$(\mathcal{E}, Q, \mathbb{D}, \eta, \mathbf{iso}) \to (\mathrm{Ch}_b(\mathcal{E}), Q_{ch}, \mathbb{D}, \eta, \mathbf{qis}),$$

where the value of Q_{ch} at a bounded chain complex X is the abelian group of pairs (ξ, φ), for $\xi \in Q(X_0)$ with $d_1^\bullet(\xi) = 0 \in Q(X_1)$, and $\varphi : X \to \mathbb{D}(X)$ a symmetric form satisfying $\varphi_0 = \rho(\xi)$ (under pointwise addition). We denote by τ and ρ also the corresponding transfer and restriction maps

$$\mathrm{Hom}_{\mathrm{Ch}_b(\mathcal{E})}(X, X) \xrightarrow{\tau_X} Q_{ch}(X) \xrightarrow{\rho_X} \mathrm{Hom}_{\mathrm{Ch}_b(\mathcal{E})}(X, X)$$

for a bounded chain complex X. The exact category $\mathrm{Ch}_b(\mathcal{E})$ has a canonical complicial structure given by an extension of the $\mathrm{Proj}_{\mathbb{Z}}$-action on $\mathcal{E}$ arising from additivity, with corresponding Frobenius exact structure consisting of degreewise split monomorphisms and epimorphisms, and Frobenius equivalences the chain homotopy equivalences; see §A.3. The construction of 3.1 then gives rise to a functor $Q_{ch}^{\Delta^\bullet} : \mathrm{Ch}_b(\mathcal{E})^{op} \to s\mathcal{A}b$ sending chain homotopies to simplicial homotopies, participating in a diagram of functors and natural transformations



where $c : \mathcal{A}b \to s\mathcal{A}b$ is the inclusion of discrete simplicial abelian groups.

Lemma 3.4.1. η_0 *is an isomorphism, as is the whiskered transformation*

$$c \circ Q \overset{c\eta_0}{\Rightarrow} c \circ Q_{ch} \circ \iota_{\mathcal{E}} \overset{(\eta_1)_{\iota_{\mathcal{E}}}}{\Rightarrow} Q_{ch}^{\Delta^\bullet} \circ \iota_{\mathcal{E}}.$$

Proof. The first statement follows from the definition of Q_{ch}; for the second, for an object x of $\mathcal{E}$, consider the map

$$cQ(x) = cQ_{ch}(x[0]) \to Q_{ch}^{\Delta^\bullet}(x[0])$$

induced by the unique map $\Delta^\bullet \to c\Delta^0$. The codegeneracies $s_n : \Delta^n x[0] \twoheadrightarrow x[0]$ induce a splitting

$$\Delta^n x[0] \cong \tilde{x}^n \oplus x[0]$$

for each n, where $\tilde{x}^n := \ker(s_n)$ is contractible. For $x \in \mathcal{E}$ and $i \in \mathbb{Z}$, write $O_i(x) := C \otimes x[i-1]$ for the disc

$$\cdots \to 0 \to x = x \to 0 \to \ldots,$$

concentrated in degrees $i, i-1$. We claim that for each $n \geq 1$ we have $\tilde{x}^n \cong \bigoplus_{1 \leq i \leq n} O_i\left(x^{\oplus \binom{n}{i}}\right)$ (we prove this in the lemma below). Assuming this for now, for each $n \geq 1$ we have

$$\begin{aligned} Q_{ch}(\Delta^n x[0]) &\cong Q_{ch}\left(x[0] \oplus \bigoplus_{1 \leq i \leq n} O_i\left(x^{\oplus \binom{n}{i}}\right)\right) \\ &\cong Q_{ch}(x[0]) \oplus Q_{ch}\left(\bigoplus_{1 \leq i \leq n} O_i\left(x^{\oplus \binom{n}{i}}\right)\right) \oplus \bigoplus_{1 \leq i \leq n} \mathrm{Hom}_{\mathrm{Ch}_b(\mathcal{E})}\left(O_i\left(x^{\oplus \binom{n}{i}}\right), \mathbb{D}(x)[0]\right), \end{aligned}$$

by quadraticity. The second summand evaluates to

$$Q_{ch}\left(\bigoplus_{1\le i\le n} O_i\left(x^{\oplus\binom{n}{i}}\right)\right) \cong \bigoplus_{1\le i\le n} Q_{ch}\left(O_i\left(x^{\oplus\binom{n}{i}}\right)\right) \oplus \bigoplus_{1\le i<j\le n} \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}\left(O_i\left(x^{\oplus\binom{n}{i}}\right), \mathbb{D}\left(O_j\left(x^{\oplus\binom{n}{j}}\right)\right)\right) = 0,$$

since

$$\mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}(O_i(y), \mathbb{D}(O_j(z))) = \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}(O_i(y), O_{1-j}(\mathbb{D}(z))) = \begin{cases} \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(z)), & j = -i, -i+1, \\ 0, & \text{else,} \end{cases}$$

for each $y, z \in \mathcal{E}$, and

$$\begin{aligned} Q_{ch}(O_i(y)) &= \{(\xi, \varphi) \mid \xi \in Q(O_i(y)_0), \varphi \in \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}(O_i(y), \mathbb{D}(O_i(y)))^{C_2}, \rho(\xi) = \varphi_0, d_1^\bullet(\xi) = 0\} \\ &\cong \begin{cases} Q(y), & i = 0, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Similarly, the third term evaluates to

$$\bigoplus_{1\le i\le n} \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}\left(O_i\left(x^{\oplus\binom{n}{i}}\right), \mathbb{D}(x)[0]\right) = 0,$$

since

$$\mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}\left(O_i\left(x^{\oplus\binom{n}{i}}\right), \mathbb{D}(x)[0]\right) = \begin{cases} \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}(x, \mathbb{D}(x)), & i = 0, \\ 0, & \text{else.} \end{cases}$$

The composite $Q_{ch}(x[0]) \to Q_{ch}(\Delta^n x[0]) \cong Q_{ch}(\tilde{x}^n) \oplus Q_{ch}(x[0]) \oplus \mathrm{Hom}_{\mathrm{Ch_b}(\mathcal{E})}(\tilde{x}^n, \mathbb{D}(x))$ is

$$\begin{pmatrix} Q_{ch}(\iota_0) \\ Q_{ch}(\iota_1) \\ \tau(\mathbb{D}(\iota_1)\rho(-)\iota_0) \end{pmatrix} \circ Q_{ch}(\pi_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and clearly this furnishes an isomorphism $Q(x) \cong Q_{ch}(x[0]) \cong Q_{ch}(\Delta^n x[0])$. But the above map coincides with the degeneracy $Q_{ch}(x[0]) = Q_{ch}(\Delta^0 x[0]) \to Q_{ch}(\Delta^n x[0])$, and hence the levelwise isomorphism above assembles into an isomorphism of simplicial abelian groups

$$cQ(x) \cong Q_{ch}^{\Delta^\bullet}(x[0]).$$

□

Lemma 3.4.2. *For each* $n \ge 1$ *we have* $\tilde{x}^n \cong \bigoplus_{1\le i\le n} O_i\left(x^{\oplus\binom{n}{i}}\right)$.

Proof. This follows from the analogous statement in $\mathscr{C}_{\mathbb{Z}}$, i.e. that

$$\tilde{\Delta}^n := \ker\left(N\,\mathbb{Z}[\Delta^n] \twoheadrightarrow N\,\mathbb{Z}[\Delta^0]\right) \cong \bigoplus_{1\le i\le n} O_i\left(\mathbb{Z}^{\oplus\binom{n}{i}}\right).$$

The category of finitely generated projective $\mathbb{Z}$-modules is idempotent complete and hence so is $\mathscr{C}_{\mathbb{Z}}$, so the contractible chain complex $\tilde{\Delta}^n$ splits as a finite direct sum of discs $O_i(\mathbb{Z})$, $\tilde{\Delta}^n \cong \bigoplus_{i\in S} O_i(\mathbb{Z}^{n_i})$ for some finite set $S \subset \mathbb{N}$ and $n_i \in \mathbb{N}$. In degrees $0 \le i \le n$, Δ^n is isomorphic to $\mathbb{Z}[(\Delta^n_i)^{nd}] \cong \mathbb{Z}^{\binom{n+1}{i+1}}$ and is 0 elsewhere, so by descending induction on the degree $0 \le i \le n$, we have that the i^{th} component of the direct sum is

$$O_i\left(\mathbb{Z}^{\oplus \sum_{i\le j\le n}(-1)^{j+i}\binom{n+1}{j+1}}\right) \cong O_i\left(\mathbb{Z}^{\oplus\binom{n}{i}}\right).$$

□

In the sequel we write Q for both the functor $\mathcal{E}^{op} \to \mathcal{A}b$ and its extension to $\mathrm{Ch}_b(\mathcal{E})$. By Corollary 3.2.5, $Q^{\Delta^\bullet}$ descends to a $D_{\geq 0}(\mathbb{Z})$-valued presheaf on the stable ∞-category $L_{\mathrm{Frob}}(\mathrm{Ch}_b(\mathcal{E})) = K_b(\mathcal{E})$ which is 2-excisive, and admits an essentially unique 2-excisive $D(\mathbb{Z})$-valued delooping which we denote $\mathfrak{P} : K_b(\mathcal{E})^{op} \to D(\mathbb{Z})$. In general, this delooping will be non-connective; we claim however that for complexes X concentrated in nonpositive degrees the natural map $Q^{\Delta^\bullet}(X) \to \mathfrak{P}(X)$, the unit of the adjunction $P_2 \dashv i$ of §3.2 is an equivalence. Observe firstly that for such X, the mapping spectrum $\mathrm{map}(X, \mathbb{D}(X))$ is connective, since $\mathbb{D}(X)$ is connective.[4] The fibre F of the map

$$\Omega Q^{\Delta^\bullet}(\Omega X) \to \Omega\tau_{\geq 0} \mathrm{Map}_\Delta(\Omega X, \mathbb{D}(\Omega X))$$

in $D(\mathbb{Z})$ sits in a long exact sequence

$$\begin{aligned} \ldots \to \pi_0 F \to \pi_1 Q^{\Delta^\bullet}(\Omega X) \to \pi_1 \mathrm{Map}(\Omega X, \mathbb{D}(\Omega X)) \to \pi_{-1} F \to \pi_0 Q^{\Delta^\bullet}(\Omega X) \to \ldots \\ \ldots \to \pi_0 \mathrm{Map}(\Omega X, \mathbb{D}(\Omega X)) \to \pi_{-2} F \to \pi_{-1} Q^{\Delta^\bullet}(X) \to 0. \end{aligned}$$

Now $\mathrm{Map}(\Omega X, \mathbb{D}(\Omega X)) \simeq \Sigma^2 \mathrm{Map}(X, \mathbb{D}(X))$ is 2-connective, and $Q^{\Delta^\bullet}(\Omega X)$ is 1-connective since $Q(\Omega X) = 0$ for coconnective X. Since $Q^{\Delta^\bullet}$ is 2-excisive as a functor to connective spectra and F is connective on complexes concentrated in negative degrees, the map $F \to Q^{\Delta^\bullet}(X)$ is a π_*-isomorphism and hence an equivalence of spectra, and in the sequential colimit, we obtain $Q^{\Delta^\bullet}(X) \simeq (P_2 Q^{\Delta^\bullet})(X)$ for coconnective complexes X. In particular, $\mathfrak{P}|_{\mathcal{E}^{op}} \simeq HQ$, for $H : \mathcal{A}b \to D(\mathbb{Z})$ the Eilenberg-MacLane embedding.

Corollary 3.4.3. $\mathfrak{P} : K_b(\mathcal{E})^{op} \to D(\mathbb{Z})$ *is the essentially unique quadratic extension of* $Q : \mathcal{E}^{op} \to \mathcal{A}b$. *Moreover, the connective cover* $\tau_{\geq 0}\mathfrak{P}(X)$ *satisfies*

$$\tau_{\geq 0}\mathfrak{P}(X) \simeq \operatorname*{colim}_{J^{op}_{X,\mathrm{Frob}}} Q$$

in $D_{\geq 0}(\mathbb{Z})$, *and the underlying loopspace* $\Omega^\infty\mathfrak{P}(x)$ *satisfies*

$$\Omega^\infty\mathfrak{P}(X) \simeq \operatorname*{colim}_{J^{op}_{X,\mathrm{Frob}}} Q$$

in $\mathcal{S}$.

Proof. By above, $\mathfrak{P}$ is a quadratic extension of HQ; uniqueness follows from [BGMN22, Th. 2.19]. □

Remark 3.4.4 (Cf. [CDH+I, Rem. 4.2.19]). It can be shown that upon restricting to coconnective chain complexes, the derived functor of Q is given by the Dold-Puppe non-additive derived functor [DP58]

$$\mathrm{Ch}_{b,\leq 0}(\mathcal{E})^{op} \hookrightarrow (\mathbb{\Delta}\mathcal{E})^{op} \simeq \mathbb{\Delta}^{op}\mathcal{E}^{op} \to s\mathcal{A}b \to D(\mathbb{Z}),$$

where the first functor is furnished by Dold-Kan, and is fully faithful in general (and upon replacing the source with all connective chain complexes, an equivalence for $\mathcal{E}$ weakly idempotent complete; see for instance [HA, Th. 1.2.3.7]). A computation shows that for coconnective X there is an equivalence

$$Q^{\Delta^\bullet}(X) \simeq \operatorname*{colim}_{[n]\in\mathbb{\Delta}^{op}} Q(X^n),$$

for $\lim_{[n]\in\mathbb{\Delta}} X^n \simeq X$ a cosimplicial resolution of X in $K_b(\mathcal{E})$, and so for $X \in \mathrm{Ch}_{b,\leq 0}(\mathcal{E})$, we recover

$$\mathfrak{P}(X) \simeq Q^{\Delta^\bullet}(X).$$

[4] This is a consequence of the natural weight structure on $K_b(\mathcal{E})$; see [HS25, Ex. 3.1.3(i)].

Passage to the derived ∞-category of $\mathcal{E}$ is then given by the Verdier projection $\pi : K_b(\mathcal{E}) \to D_b(\mathcal{E})$ at the subcategory of acyclics $Ac_b(\mathcal{E})$; left Kan extending $Ϙ$ along π, we obtain a hermitian ∞-category $(D_b(\mathcal{E}), \pi_! Ϙ)$.

Theorem 3.4.5. *For* $(\mathcal{E}, Q, \mathbb{D}, \eta)$ *an exact form category, the pair* $(D_b(\mathcal{E}), \pi_! Ϙ)$ *is the data of a Poincaré ∞-category. Moreover, the derived functor* $\pi_! Ϙ$ *satisfies*

$$\tau_{\geq 0} \pi_! Ϙ(X) \simeq \operatorname*{colim}_{J_X^{op}} Q$$

in $D_{\geq 0}(\mathbb{Z})$*, and*

$$\Omega^\infty \pi_! Ϙ(X) \simeq \operatorname*{colim}_{J_X^{op}} Q$$

in $\mathcal{S}$.

Proof. This follows from Theorem 3.2.9 (and Remark 3.2.10), in light of the fact that the duality on $\mathcal{E}$ induces a duality on $Ch_b(\mathcal{E})$, which descends to $K_b(\mathcal{E})$ and $D_b(\mathcal{E})$. □

4 Grothendieck-Witt spaces

In this section we construct a natural equivalence

$$\mathcal{GW}(\mathcal{E}, Q, w) \xrightarrow{\simeq} \mathcal{GW}(L_w(\mathcal{E}), \mathbf{R}Q),$$

for $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ a complicial exact form category with weak equivalences, and associated Poincaré category $(L_w(\mathcal{E}), \mathbf{R}Q)$. The argument builds on that of [CDH+I, App. B.2]: since the Grothendieck-Witt space in both the classical and Poincaré setting is constructed as the fibre of a map between realisations of certain simplicial spaces, it is enough to exhibit a levelwise equivalence between the latter. The comparison then reduces to an analysis of the underlying infinite loopspace of the right derived quadratic functor $\mathbf{R}Q$ of the previous section. We begin in 4.1 with a comparison of the spaces of nondegenerate forms associated to an exact form category and to a Poincaré ∞-category, which in 4.2 we extend to the hermitian $S_\bullet$- and $\mathcal{Q}_\bullet$-constructions. The comparison then follows straightforwardly in 4.3.

4.1 Moduli spaces of Poincaré objects

Given a complicial exact form category with weak equivalences $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$, write $(L_w(\mathcal{E}), \mathbf{R}Q)$ for the associated Poincaré ∞-category of §3. There is a functor $w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w) \to \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q)$ induced by naturality of unstraightening: the localisation $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ induces a map of right fibrations

$$\begin{array}{ccc} \int_{w\mathcal{E}} \gamma^* \mathbf{R}Q_{\mathrm{nd}} & \longrightarrow & \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q) \\ \downarrow & & \downarrow \\ w\mathcal{E} & \xrightarrow{\gamma} & L_w(\mathcal{E})^{\simeq}, \end{array}$$

where the left-hand vertical arrow is classified by the restriction $\gamma^* \mathbf{R}Q_{\mathrm{nd}} : w\mathcal{E}^{\mathrm{op}} \to \mathcal{S}$, and composing with the map in $\mathrm{RFib}(\mathcal{E})$ induced by the natural transformation $Q_{\mathrm{nd}} \Rightarrow \gamma^* \mathbf{R}Q_{\mathrm{nd}}$ then gives a map

$$w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w) \to \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q).$$

Remark 4.1.1. Let $(\mathcal{C}, w)$ an ∞-category with weak equivalences and fibrations, and write $\gamma : \mathcal{C} \to L_w(\mathcal{C})$ for the Dwyer-Kan localisation. Then by [Cis19, Th. 7.2.8] there is a natural equivalence of spaces

$$\operatorname*{colim}_{z \in J_x^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(z, y) \simeq \mathrm{Map}_{L_w(\mathcal{C})}(\gamma(x), \gamma(y)),$$

for each $x, y \in \mathcal{C}$, where the colimit is indexed by the (opposite of the) subcategory J_x spanned by trivial fibrations $z \xrightarrow{\sim} x$ in w with target x. Moreover, from [CDH+II, Prop. B.2.6] this restricts to the following for maps in $L_w(\mathcal{E})^{\simeq}$:

$$\operatorname*{colim}_{z \in J_x^{\mathrm{op}}} \mathrm{Map}_{w\mathcal{C}}(z, y) \simeq \mathrm{Map}_{L_w(\mathcal{C})^{\simeq}}(\gamma(x), \gamma(y)).$$

This is true for a general ∞-category with weak equivalences and fibrations, but for $(\mathcal{E}, w)$ a complicial exact category with weak equivalences, one can argue (quite messily) as follows: for $x, y \in \mathcal{E}$ write $\mathrm{Map}^w_\Delta(x, y)$ for the pullback in simplicial sets

$$\begin{array}{ccc} \mathrm{Map}^w_\Delta(x,y) & \longrightarrow & \mathrm{Hom}_{w\underline{\mathcal{E}}}(x,y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_\Delta(x,y) & \longrightarrow & \mathrm{Hom}_{\underline{\mathcal{E}}}(x,y), \end{array} \tag{4.1}$$

where $\mathrm{Hom}_{\underline{\mathcal{E}}}(x, y) = \pi_0 \mathrm{Map}_\Delta(x, y)$, and we write $w\underline{\mathcal{E}} \subset \underline{\mathcal{E}} = \mathrm{Ho}(\mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))$ for the essential image of $w\mathcal{E} \subset \mathcal{E}$ under the canonical functor to the homotopy category. Note that the pullback $\mathrm{Map}^w_\Delta(x, y)$ is simply given levelwise by $[n] \mapsto \mathrm{Map}_w(\Delta^n x, y)$, and so by Proposition 3.1.7 is equivalent to $\operatorname{colim}_{\mathrm{J}^{\mathrm{op}}_{x,\mathrm{Frob}}} \mathrm{Map}_w(-, y)$. Since the right vertical map is a Kan fibration in the classical model structure on sSet, the square (4.1) localises to a cartesian square of spaces[1]. Taking filtered colimits over the subcategory $\mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}} \subset \mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})^{\mathrm{op}}_{/x}$ spanned by the essential image of $\mathrm{J}^{\mathrm{op}}_x \subset (\mathcal{E}^{\mathrm{op}})_{/x}$ under the localisation $\mathcal{E} \to \mathrm{L}_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})$, we obtain a cartesian square

$$\begin{array}{ccc} \operatorname{colim}_{z\in \mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}}} \mathrm{Map}^w_\Delta(z,y) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{L}_w(\mathcal{E}))^\simeq}(x,y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{L}_w(\mathcal{E})}(x,y) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{L}_w(\mathcal{E}))}(x,y) \end{array}$$

where the identifications of the mapping sets on the right follow from the classical description of maps in a localisation in the presence of a right calculus of fractions: for each trivial egression $s : z \overset{\sim}{\twoheadrightarrow} x$, the maps $\mathrm{Hom}_{\underline{\mathcal{E}}}(\overline{z}, \overline{y}) \to \pi_0 \mathrm{Hom}_{\mathrm{L}_w(\mathcal{E})}(\gamma(x), \gamma(y))$, $f \mapsto fs^{-1}$ induce an isomorphism of sets

$$\pi_0 \mathrm{Map}_{\mathrm{L}_w(\mathcal{E})}(\gamma(x), \gamma(y)) \simeq \operatorname{colim}_{z\in \mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}}} \pi_0 \mathrm{Hom}_{\underline{\mathcal{E}}}(\overline{z}, \overline{y}), \tag{4.2}$$

where we write $\overline{y}$ for the image of y in $\underline{\mathcal{E}}$. Each equivalence $\gamma(x) \to \gamma(y)$ is accordingly represented by a span $x \overset{\sim}{\twoheadleftarrow} z \overset{f}{\to} y$, with $\gamma(f)$ an equivalence by 2-of-3, so that f lies in the saturation of w, and the equivalence $\gamma(x) \to \gamma(y)$ is in the image of the map

$$\operatorname{colim}_{z\in \mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}}} \mathrm{Hom}_{w\underline{\mathcal{E}}}(\overline{z}, \overline{y}) \to \pi_0 \mathrm{Map}_{\mathrm{L}_w(\mathcal{E})}(\gamma(x), \gamma(y)).$$

Since this map is monic, as a filtered colimit of monomorphisms, (4.2) restricts to an isomorphism

$$\pi_0 \mathrm{Map}_{\mathrm{L}_w(\mathcal{E})^\simeq}(\gamma(x), \gamma(y)) \simeq \operatorname{colim}_{z\in \mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}}} \pi_0 \mathrm{Hom}_{w\underline{\mathcal{E}}}(\overline{z}, \overline{y}),$$

and we have a natural equivalence

$$\mathrm{Map}_{\mathrm{L}_w(\mathcal{E})^\simeq}(\gamma(x), \gamma(y)) \simeq \operatorname{colim}_{\mathrm{L}_{\mathrm{Frob}}(\mathrm{J}^{\mathrm{op}}_x)} \mathrm{Map}^w_\Delta(z, y) \simeq \operatorname{colim}_{z\in \mathrm{L}_{\mathrm{Frob}}(\mathrm{J}_x)^{\mathrm{op}}} \operatorname{colim}_{\mathrm{J}^{\mathrm{op}}_{z,\mathrm{Frob}}} \mathrm{Map}_w(-, y) \simeq \operatorname{colim}_{\mathrm{J}^{\mathrm{op}}_x} \mathrm{Map}_w(-, y).$$

Lemma 4.1.2. *Suppose given a complicial exact form category with weak equivalences* $(\mathcal{E}, w, Q, D, \eta)$*, with derived Poincaré catgeory* $(\mathrm{L}_w(\mathcal{E}), \Omega)$*. Then there is an equivalence of spaces*

$$\Omega_{\mathrm{nd}}(x) \simeq \operatorname{colim}_{\mathrm{J}^{\mathrm{op}}_x} Q_{\mathrm{nd}},$$

for each $x \in \mathcal{E}$.

[1]Here, we use that the Kan-Quillen model structure on simplicial sets is proper, so that the strict pullback of a cospan with one leg a Kan fibration is a homotopy pullback; see [GJ09, Cor. II.9.6].

For the proof, we will need to avail ourselves of the following criterion. Suppose I is a small ∞-category, and we are given a pullback square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ Y & \longrightarrow & Z \end{array}$$

in $\mathrm{Fun}(I, \mathcal{S})$. Suppose that $\tau : X \Rightarrow Z$ in $\mathrm{Fun}(I, \mathcal{S})$ is **equifibred**, i.e. for each map $f : i \to j$ in I, the square

$$\begin{array}{ccc} X(i) & \xrightarrow{X(f)} & X(j) \\ \downarrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \tau_j} \\ Z(i) & \xrightarrow{Z(f)} & Z(j) \end{array}$$

is cartesian. Then by a result of Rezk [Rez19], the square

$$\begin{array}{ccc} \mathrm{colim}_I W & \longrightarrow & \mathrm{colim}_I X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{colim}_I \tau} \\ \mathrm{colim}_I Y & \longrightarrow & \mathrm{colim}_I Z \end{array}$$

is cartesian in $\mathcal{S}$.

Proof of Lemma 4.1.2. Recall that Q_{nd} and $\mathcal{Q}_{nd}$ are defined pointwise respectively by the cartesian squares of spaces

$$\begin{array}{ccc} Q_{nd}(x) & \hookrightarrow & Q(x) \\ \downarrow & & \downarrow{\scriptstyle \rho_x} \\ \mathrm{Hom}_{w\mathcal{E}}(x, \mathbb{D}(x)) & \hookrightarrow & \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)), \end{array} \qquad \begin{array}{ccc} \mathcal{Q}_{nd}(x) & \hookrightarrow & \Omega^\infty \mathcal{Q}(x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{L_w(\mathcal{E})^\simeq}(x, \mathbb{D}(x)) & \hookrightarrow & \mathrm{Map}_{L_w(\mathcal{E})}(x, \mathbb{D}(x)). \end{array}$$

For $x \in \mathcal{E}$, we claim that the map of diagrams

$$J_x^{op} \to \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{S}et) \to \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{S}), \quad (y \overset{\sim}{\twoheadrightarrow} x) \mapsto (\mathrm{Hom}_{w\mathcal{E}}(y, \mathbb{D}(y)) \subset \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(y)))$$

is equifibred. Indeed, for $f : y \overset{\sim}{\to} z$ a map over x (necessarily a weak equivalence), the square

$$\begin{array}{ccc} \mathrm{Hom}_{w\mathcal{E}}(z, \mathbb{D}(z)) & \hookrightarrow & \mathrm{Hom}_{\mathcal{E}}(z, \mathbb{D}(z)) \\ \downarrow{\scriptstyle f^* \mathbb{D}(f)_*} & & \downarrow{\scriptstyle f^* \mathbb{D}(f)_*} \\ \mathrm{Hom}_{w\mathcal{E}}(y, \mathbb{D}(y)) & \hookrightarrow & \mathrm{Hom}_{\mathcal{E}}(y, \mathbb{D}(y)), \end{array}$$

is cartesian by 2-of-3. Accordingly, by Rezk's equifibration criterion the square

$$\begin{array}{ccc} \mathrm{colim}_{J_x^{op}} Q_{nd} & \longrightarrow & \mathrm{colim}_{J_x^{op}} Q \\ \downarrow & & \downarrow \\ \mathrm{colim}_{J_x^{op}} \mathrm{Hom}_{w\mathcal{E}}(-, \mathbb{D}(-)) & \longrightarrow & \mathrm{colim}_{J_x^{op}} \mathrm{Hom}_{\mathcal{E}}(-, \mathbb{D}(-)) \end{array}$$

is cartesian, and this naturally identifies with the square

$$\begin{array}{ccc} \mathcal{Q}_{nd}(x) & \longrightarrow & \Omega^\infty \mathcal{Q}(x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{L_w(\mathcal{E})^\simeq}(x, \mathbb{D}(x)) & \longrightarrow & \mathrm{Map}_{L_w(\mathcal{E})}(x, \mathbb{D}(x)). \end{array}$$

□

The following generalises the argument of [CDH+II, Prop. B.2.6].

Proposition 4.1.3. *For* $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ *a complicial exact form category with weak equivalences, the functor* $\mathrm{wQuad}(\mathcal{E}, Q, w) \to \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q)$ *induces an equivalence upon realisation, which is moreover natural in duality-preserving form functors between complicial exact form categories with weak equivalences.*

Proof. Consider the square

$$\begin{array}{ccc} \mathrm{wQuad}(\mathcal{E}, Q, w) & \longrightarrow & \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q) \\ \downarrow & & \downarrow \\ w\mathcal{E} & \xrightarrow{\quad\gamma\quad} & L_w(\mathcal{E})^{\simeq}, \end{array} \tag{4.3}$$

where each vertical map is a right fibration. By [HTT, Cor. 3.3.4.6] the homotopy type of the total space of a right fibration is given by the colimit in spaces over the classified functor, and so we have an induced map

$$\operatorname*{colim}_{w\mathcal{E}^{\mathrm{op}}} Q_{\mathrm{nd}} \to \operatorname*{colim}_{L_w(\mathcal{E})^{\simeq,\mathrm{op}}} \mathbf{R}Q_{\mathrm{nd}}.$$

The functor $w\mathcal{E}^{\mathrm{op}} \to L_w(\mathcal{E})^{\simeq,\mathrm{op}}$ is cofinal by [Cis19, Cor. 7.6.9], and so it suffices to consider the map

$$\operatorname*{colim}_{w\mathcal{E}^{\mathrm{op}}} Q_{\mathrm{nd}} \to \operatorname*{colim}_{w\mathcal{E}^{\mathrm{op}}} \gamma^* \mathbf{R}Q_{\mathrm{nd}}$$

induced by the natural transformation $Q_{\mathrm{nd}} \Rightarrow \gamma^* \mathbf{R}Q_{\mathrm{nd}}$. Lemma 4.1.2 yields an identification of the target with $\operatorname*{colim}_{w\mathcal{E}^{\mathrm{op}}} \mathcal{Q}_{\mathrm{nd}}$, and we are done by Proposition A.3.24. The naturality statement follows from the fact that wQuad and Pn are functorial in duality-preserving functors in the respective settings of exact form categories with weak equivalences and Poincaré categories, and the naturality of (total) localisations. □

4.2 The hermitian $S_\bullet$-construction

We wish to upgrade Proposition 4.1.3 to a levelwise equivalence of simplicial spaces

$$([s] \mapsto |\mathrm{wQuad}(S_{2s+1}(\mathcal{E}), Q_{2s+1}, w)|) \simeq \mathrm{Pn}(\mathcal{Q}_\bullet(L_w(\mathcal{E})), \mathbf{R}Q_\bullet).$$

Since by 3.3 the construction of the derived Poincaré category is functorial, it will suffice to exhibit the latter as the (levelwise) derived Poincaré category of the former form category.

Remark 4.2.1. Since the action of $\mathscr{C}(\mathbb{Z})$ on $\mathcal{E}$ is bi-exact, for each $n \geq 0$ the pointwise complicial structure on $\mathrm{Fun}(\mathrm{Ar}(\Delta^n), \mathcal{E})$ preserves the subcategory $S_n(\mathcal{E})$, and the tuple $(S_n(\mathcal{E}), Q_n, \mathbb{D}_n, \eta_n, w)$ is the data of a complicial exact form category with weak equivalences.

Write $S_n(\mathcal{C})$ for the ∞-categorical $S_\bullet$-construction on a stable ∞-category $\mathcal{C}$, i.e. for the subcategory of functors $X : \mathrm{Ar}(\Delta^n) \to \mathcal{C}$ with $X_{i,i} \simeq 0$ for each $0 \leq i \leq n$ and $X_{ij} \to X_{ik} \to X_{jk}$ a fibre sequence in $\mathcal{C}$ for each triple $0 \leq i \leq j \leq k \leq n$. Since congressions in $\mathcal{E}$ localise to fibre sequences in $L_w(\mathcal{E})$, the localisation $\mathcal{E} \to L_w(\mathcal{E})$ induces a functor

$$S_n(\mathcal{E}) \to S_n(L_w(\mathcal{E})),$$

which moreover exhibits $S_n(L_w(\mathcal{E}))$ as the localisation of $S_n(\mathcal{E})$ at the pointwise weak equivalences, by [CDH+I, Lem. B.2.5]. Write

$$(\mathbf{R}Q)_n : S_n(L_w(\mathcal{E}))^{\mathrm{op}} \to D(\mathbb{Z}), \quad X \mapsto \lim_{i+j \leq n} \mathbf{R}Q(X_{ij}),$$

where the limit is taken over the (opposite of the) subposet of $\mathrm{Ar}(\Delta^n)$ on those objects (i,j) with $i+j \leq n$, and $\mathbf{R}(Q_n)$ for the 2-excisive (right) derived functor of the functor $Q_n : S_n(\mathcal{E})^{\mathrm{op}} \to \mathcal{A}b$ of 2.3 with respect to the pointwise weak equivalences in $S_n(\mathcal{E})$.

Remark 4.2.2. The inclusion

$$\mathcal{T}_n \overset{\iota_n}{\hookrightarrow} \{(i \leq j) \in \mathrm{Ar}(\Delta^n) \mid i+j \leq n\}, \tag{4.4}$$

for $\mathcal{T}_n$ the subposet spanned by $(i \leq j)$ with $i+j = n-1, n$, is cofinal, by an application of Quillen's theorem A: the slice category $((i \leq j) \downarrow \iota_n)$ has objects $(i \leq j) \to (i' \leq j')$, where $i'+j' \in \{n-1, n\}$., and since $\mathcal{T}_n$ is a poset, this is equivalent to the subcategory of $\mathcal{T}_n$ on pairs $(i' \leq j')$ with $i \leq i'$ and $j \leq j'$, which as a (downwards-closed) subtree of $\mathcal{T}_n$ is contractible.

The functor

$$S_n(\mathcal{E})^{\mathrm{op}} \xrightarrow{\gamma} S_n(L_w(\mathcal{E}))^{\mathrm{op}} \xrightarrow{(\mathbf{R}Q)_n} \mathcal{S}p$$

receives a natural transformation from Q_n via the composite

$$Q_n(X) \hookrightarrow \lim_{\mathcal{T}_n} (Q \circ X^{\mathrm{op}}) \to \lim_{\mathcal{T}_n} (\mathbf{R}Q \circ \gamma(X)^{\mathrm{op}})$$

(where the first limit is taken in abelian groups, or equivalently connective spectra, and the second in spectra), factoring uniquely through a map $\mathbf{R}(Q_n) \to (\mathbf{R}Q)_n$ which we show in Proposition 4.2.5 to be an equivalence for n odd (i.e. in those degrees corresponding to the subdivided $S_\bullet$-construction).
For $x \in \mathcal{E}$ and $1 \leq i \leq n+1$, write $s_i^n(x)$ for the image of x under the map $\mathcal{E} = S_1(\mathcal{E}) \to S_n(\mathcal{E})$ induced by the surjection $[n] \to [1]$ sending $0 \leq j < i$ to 0, and $i \leq j \leq n$ to 1, so that $s_i^n(x)$ is the diagram given by

$$s_i^n(x)_{jk} = \begin{cases} x, & 0 \leq j < i, n \geq k \geq i, \\ 0, & \text{else.} \end{cases}$$

For instance, $s_2^4(x)$ has shape:

$$\begin{array}{ccccccc} 0 & \longrightarrow & x & = & x & = & x \\ & & \| & & \| & & \| \\ & & x & = & x & = & x \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & \longrightarrow & 0 \\ & & & & & & \downarrow \\ & & & & & & 0. \end{array} \tag{4.5}$$

Note that $\mathbb{D}(s_i^n(x)) = s_{n+1-i}^n(\mathbb{D}(x))$. Recall from 2.3 that for each $X \in S_n(\mathcal{E})$, we have an injection $Q_n(X) \to \lim_{\mathcal{T}_n} Q \circ X^{\mathrm{op}}$. Recall also that we write $\mathcal{T}_n$ for the subposet on pairs $(i \leq j)$ with $i+j = n-1$ or n.

Lemma 4.2.3. *For each odd* $n \geq 1$, *the map* $Q_n(s_i^n(x)) \to \lim_{\mathcal{T}_n} Q \circ s_i^n(x)^{\mathrm{op}}$ *is an isomorphism, for* $s_i^n(x)$ *considered an object of* $S_n(\mathcal{E})$.

Remark 4.2.4. Note that the statement of the lemma is false for even n: indeed

$$\lim_{\mathcal{T}_4} Q \circ s_2^4(x)^{\mathrm{op}} = \lim(Q(x = x = x)) = Q(x),$$

but a map $\varphi : s^4_2(x) \to \mathbb{D}(s^4_2(x))$ is necessarily zero in view of the subdiagram

$$\begin{array}{ccc} x = s^4_2(x)_{14} & \xrightarrow{\varphi} & \mathbb{D}(s^4_2(x))_{14} = \mathbb{D}(x) \\ \downarrow & & \| \\ 0 = s^4_2(x)_{24} & \longrightarrow & \mathbb{D}(s^4_2(x))_{24} = \mathbb{D}(x). \end{array}$$

We shall only need the above for the comparison of quadratic functors on the subdivided $S_\bullet$-construction, for which odd n suffice.

Proof of lemma 4.2.3. This is a matter of tedious verification. Suppose x is not a zero object. Then we need to show that any compatible collection of points $(\xi_{jk})_{(j\le k)\in\mathcal{T}_n} \in \prod_{(j\le k)\in\mathcal{T}_n} Q(s^n_i(x)_{jk})$ induces a map of diagrams $s^n_i(x) \to s^n_{n+1-i}(\mathbb{D}(x))$. Since such a map is prescribed by $\rho(\xi_{0n}) : s^n_i(x)_{0n} = x \to \mathbb{D}(x) = \mathbb{D}(s^n_i(x))$, we need only check that the extension of $\rho(\xi_{0n})$ to a map of diagrams imposes no constraint on ξ_{0n}; for this, it suffices to check that the limit $\lim_{\mathcal{T}_n}(Q\circ s^n_i(x)^{op})$ is zero precisely when $\mathrm{Map}_{S_n(\mathcal{E}_{\mathrm{Frob}}}(s^n_i(x), \mathbb{D}(s^n_i(x))) = 0$.

Now $\lim_{\mathcal{T}_n}(Q \circ s^n_i(x)^{op})$ vanishes if and only if the element $(j \le k) \in \mathcal{T}_n$ with minimal j and maximal k for which $s^n_i(x)_{jk} = 0$ has $k = n - j$, since in this case we have a cofinal subdiagram of $s^n_i(x)$ (mapping under Q to a final subdiagram)

$$\begin{array}{ccc} x = s^n_i(x)_{j-1,n-j} & \longrightarrow & s^n_i(x)_{j-1,n-j+1} = x, \\ \downarrow & & \\ 0 = s^n_i(x)_{j,n-j}, & & \end{array}$$

such that by definition of $s^n_i(x)$ we have $j - 1 < i$, $n - j \ge i$, and $j \ge i$, i.e. $i = j$. Since n is odd, then since $j - 1 < n - j$ and both sides have the same parity, we have also $j < n - j$, and $i < \frac{n}{2}$. One checks that $s^n_i(x)$ is nonzero for each $(j \le k) \in \mathcal{T}_n$ if and only if $i = \frac{n+1}{2}$. If the element $(j \le k) \in \mathcal{T}_n$ with minimal j and maximal k for which $s^n_i(x)_{jk} = 0$ has $k = n - j - 1$, we have a cofinal subdiagram of the form

$$\begin{array}{ccc} 0 = s^n_i(x)_{j,n-j-1} & \longrightarrow & s^n_i(x)_{j,n-j} = x, \\ \downarrow & & \\ 0 = s^n_i(x)_{j+1,n-j-1} & & \end{array}$$

so $j < i$, $n - j \ge i$, and $n - j - 1 < i$, i.e. $i = n - j$, so that since again $j < n - j - 1$, we have $i > \frac{n+1}{2}$, and so

$$\lim_{\mathcal{T}_n} Q \circ s^n_i(x) = \begin{cases} Q(x), & i \ge \frac{n+1}{2}, \\ 0, & i < \frac{n+1}{2}. \end{cases}$$

Now a map $\varphi : s^n_i(x) \to \mathbb{D}(s^n_i(x)) = s^n_{n+1-i}(\mathbb{D}(x))$ is determined by the component $\varphi_{0n} : x \to \mathbb{D}(x)$ as in 2.3, and accordingly is prescribed to be φ_{0n} when both the domain and target are nonzero, and 0 otherwise. φ_{0n} is zero if and only if φ has a subdiagram of the form

$$\begin{array}{ccc} x & \xrightarrow{\varphi_{0n}} & \mathbb{D}(x) \\ \downarrow & & \| \\ 0 & \longrightarrow & \mathbb{D}(x) \end{array}$$

or

$$\begin{array}{ccc} x & \xrightarrow{\varphi_{0n}} & 0 \\ \| & & \downarrow \\ x & \longrightarrow & \mathbb{D}(x), \end{array}$$

and one checks that each of these are equivalent to the stipulation that $i < \frac{n+1}{2}$. $\square$

Proposition 4.2.5. *For each odd* $n \geq 1$ *the canonical map* $\mathbf{R}(Q_n) \to (\mathbf{R}Q)_n$ *is an equivalence of quadratic functors.*

Proof. An outline of the proof is as follows: we show that $\mathbf{R}(Q_n) \to (\mathbf{R}Q)_n$ is an equivalence on a class of diagrams in $S_n(L_w(\mathcal{E}))$ which generate this category under extensions; Lemma 4.2.6 then yields an equivalence for all diagrams. As usual, we start by considering the Frobenius structure (on $S_n(\mathcal{E})$), and derive from here. The case $n = 1$ holds by the identifications $Q_1 \cong Q$, $(\mathbf{R}Q)_1 \simeq \mathbf{R}Q$. We will use that since $S_n(\mathcal{E})$ has a pointwise complicial structure, the localisation $\gamma : S_n(\mathcal{E}) \to L_{w_{pt}}(S_n(\mathcal{E}))$ factors as the composite

$$S_n(\mathcal{E}) \to L_{\mathrm{Frob}}(S_n(\mathcal{E})_{\mathrm{Frob}}) \to L_{w_{pt}}(S_n(\mathcal{E})_{\mathrm{Frob}}) = L_{w_{pt}}(S_n(\mathcal{E})) = S_n(L_w(\mathcal{E})).$$

Write $\mathbf{R}_{\mathrm{Frob}}(Q_n)$ for the quadratic right derived functor of Q_n at the Frobenius equivalences internal to $S_n(\mathcal{E})$, so that by Corollary 3.2.5 and Proposition 3.2.6 there is an equivalence

$$\Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_n)(X) \simeq Q_n^{\Delta^\bullet}(X)$$

for each $X \in S_n(\mathcal{E})$, where the tensor with $\Delta^\bullet$ is computed pointwise. Now a diagram $X \in S_n(\mathcal{E})$ is determined by the first row $(X_{0,1} \to X_{0,2} \to \cdots \to X_{0n})$: indeed, writing $\mathscr{I} \subset \mathrm{Ar}(\Delta^n)$ for the subcategory spanned by pairs (i,j) with either $i = 0$ or $i = j$, X is the left Kan extension along the inclusion $\mathscr{I} \subset \mathrm{Ar}(\Delta^n)$ of the restriction $X|_{\mathscr{I}}$. Now there is a filtration

$$X^1 \hookrightarrow X^2 \hookrightarrow \cdots \hookrightarrow X^{n-1} \hookrightarrow X^n = X,$$

for X^k the diagram generated by the row $(X_{0,1} \to \cdots \to X_{0,k-1} \to X_{0,k} = X_{0,k} = \cdots = X_{0,k})$, and for each $1 \leq k \leq n-1$ the quotient $X^{k,k+1} := \mathrm{cofib}(X^k \hookrightarrow X^{k+1})$ satisfies

$$\left(X^{k,k+1}\right)_{ij} = \begin{cases} X_{k,k+1}, & 0 \leq i < k+1, j \geq k+1, \\ 0, & \text{else}, \end{cases}$$

i.e. $X^{k,k+1} = s^n_{k+1}(X_{k,k+1})$. We note also that $X^1 = s^n_1(X_{01})$, so that the subcategory of diagrams of the form $s^n_i(x)$ for $x \in \mathcal{E}$ generates $L_{\mathrm{Frob}}(S_n(\mathcal{E}))$ under extensions. Now from the preceding lemma,

$$\Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_n)(s^n_i(x)) = Q_n^{\Delta^\bullet}(s^n_i(x)) = \lim_{\mathcal{T}_n}(Q^{\Delta^\bullet} \circ s^n_i(x)^{\mathrm{op}}) = \lim_{\mathcal{T}_n}(\Omega^\infty \mathbf{R}_{\mathrm{Frob}} Q \circ s^n_i(x)^{\mathrm{op}}),$$

where the limit is taken in the ordinary category of simplicial abelian groups. In this case, this limit is equivalent to the homotopy limits computed in simplicial abelian groups, since each such diagram has a final subdiagram of the form

$$\begin{array}{ccc} a & \xleftarrow[\mathrm{id}_a]{} & a \\ \uparrow & & \\ 0 & & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} 0 & \longleftarrow & a, \\ \uparrow & & \\ 0 & & \end{array}$$

the limits over which agree. Accordingly, there is an equivalence

$$\Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_n)(s_i^n(x)) \simeq \Omega^\infty \lim_{\mathcal{T}_n} (\mathbf{R}_{\mathrm{Frob}} Q \circ s_i^n(x)^{\mathrm{op}})$$

induced by the identification $\Omega^\infty \mathbf{R}_{\mathrm{Frob}} Q \simeq Q^{\Delta^\bullet}$, where the limit is taken in spectra.

Now the value of the right derived functor of $\mathbf{R}_{\mathrm{Frob}}(Q_n)$ at some $X \in S_n(\mathcal{E})$ along the localisation $L_{\mathrm{Frob}}(S_n(\mathcal{E})_{\mathrm{Frob}}) \to L_{w_{\mathrm{pt}}}(S_n(\mathcal{E}))$ at the pointwise w-equivalences is computed as the (filtered) colimit over the subcategory

$$I_X \subset L_{\mathrm{Frob}}(S_n(\mathcal{E}))_{/X}$$

spanned by the pointwise w-equivalences over X. In the case $X = s_i^n(x)$ for some $x \in \mathcal{E}$, we claim that the subcategory of pointwise w-equivalences over $s_i^n(x)$ of the form $s_i^n(y) \xrightarrow{s_i^n(x)} s_i^n(x)$ for some $f : y \to x$ in w is cofinal in I_x. Indeed, consider the functors

$$\mathrm{ev}_{0i} : S_n(\mathcal{E})_{/s_i^n(x)} \to (\mathcal{E}_{\mathrm{Frob}})_{/x}, \quad (X \xrightarrow{F} s_i^n(x)) \xrightarrow{F_{0i}} (X_{0i} \to x)$$

and

$$s_i^n(-) : (\mathcal{E}_{\mathrm{Frob}})_{/x} \to S_n(\mathcal{E})_{/s_i^n(x)}, \quad (y \xrightarrow{f} x) \mapsto (s_i^n(y) \xrightarrow{s_i^n(f)} s_i^n(x)).$$

The evaluation ev_{0i} sends maps over $s_i^n(x)$ which are Frobenius equivalences internal to $S_n(\mathcal{E})$ to equivalences in $L_{\mathrm{Frob}}(\mathcal{E})_{/x}$, since it inverts pointwise Frobenius equivalences. Note that a pointwise Frobenius equivalence $s_i^n(x) \to s_i^n(y)$ is in fact a Frobenius equivalence internal to $S_n(\mathcal{E})$, since a Frobenius inverse to the underlying map $x \to y$ gives a Frobenius inverse to $s_i^n(x) \to s_i^n(y)$ (each of the constituent maps in these diagrams is either the identity or 0, so this promotes uniquely to a natural transformation of diagrams). The functor $s_i^n(-)$ sends maps over x which are Frobenius equivalences to pointwise Frobenius equivalences between objects of the form $s_i^n(y)$, which are accordingly Frobenius equivalences internal to $S_n(\mathcal{E})$. Given $X \to s_i^n(x)$ and $y \to x$, the induced map on mapping sets is the equivalence

$$\mathrm{Hom}_{S_n(\mathcal{E})_{/s_i^n(x)}}(s_i^n(y) \to s_i^n(x), X \to s_i^n(x)) \to \mathrm{Hom}_{\mathcal{E}_{/x}}(y, X_{0i})$$

which is an isomorphism since, restricting to the first row via the equivalence of categories $S_n(\mathcal{E}) \simeq \mathrm{Fun}_{\mathrm{in}}(\Delta^n, \mathcal{E})$ (where the subscripts denotes that we consider the full subcategory of the functor category on functors sending each map of Δ^n to an ingression of $\mathcal{E}$), a map $(s_i^n(y)_{00} \rightarrowtail \ldots \rightarrowtail s_i^n(y)_{0n} = y) \to (X_{00} \rightarrowtail X_{01} \rightarrowtail \ldots \rightarrowtail X_{0n})$ is determined by the component at the first nonzero coordinate of $s_i^n(y)$, which occurs at $(0, i)$ (the preceding maps having source 0): maps from the diagram $s_2^4(x)$ (4.5) as below

$$\begin{array}{ccccccc}
0 & \longrightarrow & x & = & x & = & x \\
\downarrow & & \downarrow{\scriptstyle f_{02}} & & \downarrow{\scriptstyle f_{03}} & & \downarrow{\scriptstyle f_{04}} \\
x_{01} & \longrightarrow & x_{02} & \longrightarrow & x_{03} & \longrightarrow & x_{04}
\end{array}$$

are determined by f_{02}. We accordingly have an adjoint pair $s_i^n(-) \dashv \mathrm{ev}_{0,i}$. Equipping $S_n(\mathcal{E})$ with the structure of an ∞-category of fibrant objects in which the weak equivalences are the internal Frobenius equivalences, and the fibrations the internal Frobenius egressions, and $\mathcal{E}$ with the structure of an ∞-category of fibrant objects in which weak equivalences are the Frobenius equivalences and the fibrations the Frobenius egressions, the slice categories inherit the structures of ∞-categories with weak equivalences and fibrations by declaring a map in the slice to be in these classes if its image under the forgetful functor is in the corresponding class (see [Cis19, Rem. 7.6.12]). We then obtain a pair of functors on localisations

$$\mathrm{ev}_{0n} : L_{\mathrm{Frob}}(\mathcal{E})_{/s_i^n(x)} \to L_{\mathrm{Frob}}(\mathcal{E})_{/x}, \quad L_{\mathrm{Frob}}(\mathcal{E})_{/x} \to L_{\mathrm{Frob}}(\mathcal{E})_{/s_i^n(x)},$$

and moreover the functors $\mathcal{E}_{/x} \to L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}})_{/x}$ and $S_n(\mathcal{E})_{s_i^n(x)} \to L_{\mathrm{Frob}}(S_n(\mathcal{E}))_{/s_i^n(x)}$ are localisations at the classes of equivalences described above by [Cis19, Cor. 6.3.13].
Finally, the claim is that this pair of functors is adjoint: indeed, for $\mathcal{C}$ an ∞-category and $x \in \mathcal{C}$, by [HTT, Lem. 5.5.5.12] the mapping space in the slice $\mathcal{C}_{/x}$ is computed as a pullback of maps in $\mathcal{C}$:

$$\mathrm{Map}_{\mathcal{C}}(y \xrightarrow{f} x, z \xrightarrow{g} x) \simeq \mathrm{fib}_{\{f\}}(\mathrm{Map}_{\mathcal{C}}(y,z) \xrightarrow{g_*} \mathrm{Map}_{\mathcal{C}}(y,x)).$$

Since mapping spaces in Frobenius localisations are computed as simplicial mapping spaces in the overlying complicial exact category, we see that the induced map on mapping spaces is the equivalence

$$\mathrm{fib}\left(\mathrm{Map}_{S_n(\mathcal{E})}(\Delta^\bullet \otimes s_i^n(y), X) \to \mathrm{Map}_{S_n(\mathcal{E})}(\Delta^\bullet s_i^n(y), s_i^n(x))\right) \to \mathrm{fib}\left(\mathrm{Map}_{\mathcal{E}}(\Delta^\bullet y, X_{0i}) \to \mathrm{Map}_{\mathcal{E}}(\Delta^\bullet y, x)\right).$$

The functor ev_{0i} clearly sends pointwise w-equivalences to w-equivalences, and likewise $s_i^n(-)$ sends w-equivalences to pointwise w-equivalences, so that this adjunction restricts to the subcategories $L_{\mathrm{Frob}}(I_{s_i^n(x)})$ and $L_{\mathrm{Frob}}(I_x)$. Taking the colimit indexed by opposite categories and using that $s_i^n(-)^{\mathrm{op}} \vdash \mathrm{ev}_{0i}^{\mathrm{op}}$ thus allows us to compute

$$\begin{aligned}\mathbf{R}_{w_{\mathrm{pt}}}(Q_n)(s_i^n(x)) &\simeq \underset{L_{\mathrm{Frob}}(I_x)}{\mathrm{colim}}\ \lim_{\mathcal{T}_n} (\mathbf{R}_{\mathrm{Frob}}(Q) \circ s_i^n(y)^{\mathrm{op}}) \simeq \lim_{\mathcal{T}_n}\ \underset{L_{\mathrm{Frob}}(I_x)}{\mathrm{colim}} (\mathbf{R}_{\mathrm{Frob}}(Q) \circ s_i^n(y)^{\mathrm{op}}) \\ &\simeq \lim_{\mathcal{T}_n} (\mathbf{R}_w(Q) \circ s_i^n(y)^{\mathrm{op}}).\end{aligned}$$

This equivalence is indeed induced by the canonical map $\mathbf{R}(Q_n) \to (\mathbf{R}Q)_n$, since the natural map $Q_n \to Q_n^{\Delta^\bullet}$ induces a commutative diagram for each $X \in S_n(\mathcal{E})$

$$\begin{array}{ccc} Q_n(X) & \longrightarrow & \lim_{\mathcal{T}_n}^{\mathcal{S}\mathrm{p}_{\geq 0}} (Q \circ X^{\mathrm{op}}) \\ \downarrow & & \downarrow \\ Q_n^{\Delta^\bullet}(X) & \longrightarrow & \lim_{\mathcal{T}_n}^{\mathcal{S}\mathrm{p}_{\geq 0}} (Q^{\Delta^\bullet} \circ X^{\mathrm{op}}), \end{array}$$

where the composite $Q_n(X) \to \lim_{\mathcal{T}_n}^{\mathcal{S}\mathrm{p}_{\geq 0}} (Q^{\Delta^\bullet} \circ X^{\mathrm{op}}) \to \lim_{\mathcal{T}_n}^{\mathcal{S}\mathrm{p}} (\mathbf{R}Q \circ X^{\mathrm{op}})$ induces the comparison map on connective covers for $w = w_{\mathrm{Frob}}$, and the composite $Q_n^{\Delta^\bullet}(s_i^n(x)) \to \lim_{\mathcal{T}_n}^{\mathcal{S}\mathrm{p}_{\geq 0}} (Q^{\Delta^\bullet} \circ s_i^n(x)^{\mathrm{op}})$ is the equivalence above. Since the canonical map to the derived functor is the inclusion into the colimit $R_{\mathrm{Frob}} Q \to \underset{y \xrightarrow{\simeq} x}{\mathrm{colim}}\ R_{\mathrm{Frob}} Q(y)$ by the pointwise formula for left Kan extensions, taking colimits of the above gives the comparison map $\mathbf{R}(Q_n) \to (\mathbf{R}Q)_n$.
We now have a map of quadratic functors $\mathbf{R}_{w_{\mathrm{pt}}}(Q_n) \to (\mathbf{R}_w Q)_n$ which when restricted to the class of objects $s_i^n(x)$ is an equivalence on connective covers. Since the class of diagrams $s_i^n(x)$ in $L_{\mathrm{Frob}}(S_n(\mathcal{E}))$ is closed under desuspensions (indeed, $\Omega s_i^n(x)$ in $L_{\mathrm{Frob}}(S_n(\mathcal{E}))$ is the image under the localisation functor of $\ker(P \otimes s_i^n(x) \twoheadrightarrow s_i^n(x))$[2]) and direct sums and generates $L_{\mathrm{Frob}}(S_n(\mathcal{E}))$ under extensions, the equivalence $\mathbf{R}(Q_n)(X) \simeq \lim_{\mathcal{T}_n} (\mathbf{R}Q \circ X^{\mathrm{op}})$ for $X \in L_w(S_n(\mathcal{E}))$ then follows from Lemma 4.2.6 below. □

[2] Here, for y an object of a complicial exact category, $P \otimes y$ is the 'path object' associated to y, for P the chain complex given by $0 \to \mathbb{Z} = \mathbb{Z} \to 0$ concentrated in degrees $0, -1$ (see Appendix A.3, discussion below (A.5)).

Lemma 4.2.6. *Suppose given a stable ∞-category $\mathcal{C}$ generated by a set $S \subset \mathcal{C}$ of objects under extensions (i.e., the smallest extension-closed subcategory $\mathcal{D} \subset \mathcal{C}$ containing S is $\mathcal{C}$ itself), and a natural transformation of reduced 2-excisive functors $\alpha : \Phi \Rightarrow \Psi : \mathcal{C}^{op} \to \mathcal{S}p$ which induces an equivalence on polarisations. Suppose moreover that S is closed under desuspensions and finite sums. Then if for each $x \in S$ we have $\tau_{\geq 0}\alpha_x : \tau_{\geq 0}\Phi(x) \simeq \tau_{\geq 0}\Psi(x)$ in $\mathcal{S}p_{\geq 0}$, $\alpha : \Phi \to \Psi$ is an equivalence.*

Proof. We claim firstly that the equivalences $\tau_{\geq 0}\alpha_x : \tau_{\geq 0}\Phi(x) \simeq \tau_{\geq 0}\Psi(x)$ deloop to equivalences of spectra $\alpha_x : \Phi(x) \simeq \Psi(x)$. By 2-excisivity of Φ and Ψ, for $x \in S$ we have a map of fibre sequences

$$\begin{array}{ccccc} \Phi(x) & \longrightarrow & \Omega\Phi(\Omega x) & \longrightarrow & \Omega B_\Phi(\Omega x, \Omega x) \\ \downarrow & & \downarrow & & \downarrow \\ \Psi(x) & \longrightarrow & \Omega\Psi(\Omega x) & \longrightarrow & \Omega B_\Psi(\Omega x, \Omega x), \end{array}$$

and since $\Omega x \in S$, we have by descending induction on i that $\pi_i\Phi(x) \cong \pi_i\Psi(x)$ for each $i \in \mathbb{Z}$, using the following diagram:

$$\begin{array}{ccccccccc} \pi_{i+1}\Phi(\Omega x) & \longrightarrow & \pi_{i+1}B_\Phi(\Omega x, \Omega x) & \longrightarrow & \pi_{i-1}\Phi(x) & \longrightarrow & \pi_i\Phi(\Omega x) & \longrightarrow & \pi_i B_\Phi(\Omega x, \Omega x) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ \pi_i\Omega\Psi(\Omega x) & \longrightarrow & \pi_{i+1}B_\Psi(\Omega x, \Omega x) & \longrightarrow & \pi_{i-1}\Psi(x) & \longrightarrow & \pi_i\Psi(\Omega x) & \longrightarrow & \pi_i B_\Psi(\Omega x, \Omega x). \end{array}$$

Now given an exact sequence $x \to y \to z$ in $\mathcal{C}$ with $x, z \in S$, we have by [CDH+I, Cor. 1.1.21] an equivalence of total fibres

$$\mathrm{fibt}\begin{pmatrix} \Phi(y) & \longrightarrow & \Phi(x) \\ \downarrow & & \downarrow \\ B_\Phi(y,x) & \longrightarrow & B_\Phi(x,x) \end{pmatrix} \simeq \Phi(z) \simeq \Psi(z) \simeq \mathrm{fibt}\begin{pmatrix} \Psi(y) & \longrightarrow & \Psi(x) \\ \downarrow & & \downarrow \\ B_\Psi(y,x) & \longrightarrow & B_\Psi(x,x) \end{pmatrix},$$

and accordingly a map of fibre sequences

$$\begin{array}{ccccc} \Phi(z) & \longrightarrow & \mathrm{fib}(\Phi(y) \to \Phi(x)) & \longrightarrow & \mathrm{fib}(B_\Phi(y,x) \to B_\Phi(x,x)) \\ \downarrow\simeq & & \downarrow & & \downarrow \\ \Psi(z) & \longrightarrow & \mathrm{fib}(\Psi(y) \to \Psi(x)) & \longrightarrow & \mathrm{fib}(B_\Psi(y,x) \to B_\Psi(x,x)) \end{array}$$

in which the first and third vertical arrows, and hence the second, are equivalences, and since the map $\Phi(x) \to \Psi(x)$ is an equivalence, so is the map $\Phi(y) \to \Psi(y)$. □

Propositions 4.1.3 and 4.2.5 then imply:

Corollary 4.2.7. *Let $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ be a complicial exact form category with weak equivalences. Then for each $k \geq 0$, the map*

$$\mathrm{wQuad}(S_{2k+1}(\mathcal{E}), Q_{2k+1}, w_{pt}) \to \mathrm{Pn}(S_{2k+1}(L_w(\mathcal{E})), (\mathbf{R}Q)_{2k+1})$$

induces an equivalence upon realisation.

4.3 Comparing Grothendieck-Witt spaces

We are now ready to prove the main result of the thesis.

Theorem 4.3.1. *Let* $(\mathcal{E}, Q, \mathbb{D}, \eta, w)$ *be a complicial exact form category with weak equivalences, with associated Poincaré category* $(L_w(\mathcal{E}), \mathbf{R}Q)$*. Then the localisation functor* $\mathcal{E} \to L_w(\mathcal{E})$ *induces a natural equivalence of Grothendieck-Witt spaces*

$$\mathcal{GW}(\mathcal{E}, Q) \to \mathcal{GW}(L_w(\mathcal{E}), \mathbf{R}Q).$$

Proof. Write $S^e_n(\mathcal{E}, Q, w)$ for the subdivided hermitian $S_\bullet$-construction on an exact form category with weak equivalences. Then by Proposition 4.2.7, for each $n \geq 0$ we have equivalences

$$|\,w\mathcal{Q}\mathrm{uad}(S^e_n(\mathcal{E}, Q, w))| \xrightarrow{\simeq} \mathrm{Pn}(\mathcal{Q}_n(L_w(\mathcal{E})), \mathbf{R}Q_n),$$

assembling into a levelwise equivalence of simplicial spaces

$$([s] \mapsto |\,w\mathcal{Q}\mathrm{uad}(S_{2s+1}(\mathcal{E}), Q_{2s+1})|) \simeq \mathrm{Pn}(\mathcal{Q}_\bullet(L_w(\mathcal{E})), \mathbf{R}Q_\bullet),$$

and accordingly, since the composite $wS^e_\bullet(\mathcal{E}) \to S^e_\bullet(L_w(\mathcal{E}))^\simeq \to \mathcal{Q}_\bullet(L_w(\mathcal{E}))^\simeq$ realises to an equivalence by [Cis19, Cor. 7.6.9] (since $S^e_n(\mathcal{E})$ has a canonical complicial structure for each n and is accordingly a category of fibrant objects), the induced map on fibres

$$\begin{array}{ccccc}
\mathcal{GW}(\mathcal{E}, Q, w) & \longrightarrow & |\,w\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathcal{E}, Q^e_\bullet, w)| & \longrightarrow & |wS_\bullet\mathcal{E}| \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{GW}(L_w(\mathcal{E}), \mathbf{R}Q) & \longrightarrow & |\mathrm{Pn}(\mathcal{Q}_\bullet(L_w(\mathcal{E})), \mathbf{R}Q_\bullet)| & \longrightarrow & |(\mathcal{Q}_\bullet(L_w(\mathcal{E}))^\simeq|
\end{array}$$

is an equivalence. □

In particular, we have the following.

Corollary 4.3.2. *Suppose* $(\mathcal{E}, Q, \mathbb{D}, \eta)$ *is an exact form category, with derived Poincaré category* $(D_b(\mathcal{E}), \mathfrak{P})$*; then there is a natural equivalence of Grothendieck-Witt spaces*

$$\mathcal{GW}(\mathcal{E}, Q) \to \mathcal{GW}(D_b(\mathcal{E}), \mathfrak{P}).$$

Proof. We have equivalences

$$\mathcal{GW}(\mathcal{E}, Q) \to \mathcal{GW}(\mathrm{Ch}_b(\mathcal{E}), Q, \mathbf{qis}) \to \mathcal{GW}(D_b(\mathcal{E}), \mathfrak{P}),$$

where the first is the Gillet-Waldhausen equivalence of [Sch24b], and the second follows from Theorem 4.3.1 since $(\mathrm{Ch}_b(\mathcal{E}), Q, \mathbf{qis}, \mathbb{D}, \eta)$ is a complicial exact form category with weak equivalences. □

5 Grothendieck-Witt spectra

In this section we refine the equivalence of Theorem 4.3.1 to an equivalence of Grothendieck-Witt spectra. In both the form and Poincaré-categorical setting, the Grothendieck-Witt space admits a canonical delooping into a generally non-connective spectrum GW; we first review the relevant formalisms.

5.1 Delooping the hermitian K-theory of complicial exact form categories

Suppose given a complicial exact from category with weak equivalences $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$. For any small category I, the functor category $\mathcal{E}^I := \mathrm{Fun}(I, \mathcal{E})$ inherits a pointwise complicial structure via

$$\otimes^I : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E}^I \to \mathcal{E}^I, \quad (X, F) \mapsto (i \mapsto X \otimes F(i)),$$

adjoint to $\otimes \circ (1 \times \mathrm{ev}_{\mathcal{E}}) : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E}^I \times I \to \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$, for ev_I the evaluation functor $(F, i) \mapsto F(i)$. For $I = [1]$, the arrow category $\mathrm{Ar}(\mathcal{E}) := \mathcal{E}^{[1]}$ inherits a duality

$$\mathbb{D}(x \xrightarrow{f} y) = (\mathbb{D}(y) \xrightarrow{\mathbb{D}(f)} \mathbb{D}(x)),$$

with double dual identification $\eta_f = (\eta_x, \eta_y)$. Frobenius contractible objects of $\mathrm{Ar}(\mathcal{E})$ are those maps f which are retracts of cones (on maps), i.e. admitting factorisations

$$\begin{array}{ccccc} & & \mathrm{id}_x & & \\ x & \longrightarrow & C \otimes u & \twoheadrightarrow & x \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle C\otimes g} & & \downarrow{\scriptstyle f} \\ y & \longrightarrow & C \otimes v & \twoheadrightarrow & y, \\ & & \mathrm{id}_y & & \end{array}$$

while pointwise Frobenius contractible maps are those f admitting pointwise factorisations

$$\begin{array}{ccccc} & & \mathrm{id}_x & & \\ x & \longrightarrow & C \otimes u & \twoheadrightarrow & x \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle f} \\ y & \longrightarrow & C \otimes v & \twoheadrightarrow & y. \\ & & \mathrm{id}_y & & \end{array}$$

There are (generally strict) inclusions $w_{\mathrm{Frob}} \subset w_{\mathrm{Frob}_{\mathrm{pt}}} \subset \mathrm{Ar}(\mathcal{E})$, where the former is the class of Frobenius equivalences internal to $\mathrm{Ar}(\mathcal{E})$, and the latter the class of maps whose image under the source and target functors are Frobenius equivalences in $\mathcal{E}$. Write also $w_{\mathrm{pt}} \subset w_{\mathrm{cone}}$ for the classes of maps which are pointwise in w, resp. such that the induced map on cones $C(f) \to C(g)$ is a weak equivalence in $\mathcal{E}$, where the inclusion arises by commuting colimits: given a pointwise weak equivalence $\alpha : f \to f'$, the cone of the induced map $g : C(f) \to C(f')$ on cones is a pushout of acyclics, and hence acyclic:

$$\begin{aligned} C(g) &\simeq \operatorname{colim}(C \otimes C(f) \twoheadleftarrow C(f) \xrightarrow{g} C(f')) \\ &\simeq \operatorname{colim}(C \otimes C \otimes x \twoheadleftarrow C \otimes x \to C \otimes y) \twoheadleftarrow \operatorname{colim}(C \otimes x \twoheadleftarrow x \to y) \to \operatorname{colim}(C \otimes x' \twoheadleftarrow x' \to y') \\ &\simeq \operatorname{colim}(C \otimes C \otimes x \twoheadleftarrow C \otimes x \to C \otimes x') \twoheadleftarrow \operatorname{colim}(C \otimes x \twoheadleftarrow x \xrightarrow{\alpha_x} x') \to \operatorname{colim}(C \otimes y \twoheadleftarrow y \xrightarrow{\alpha_y} y') \\ &\simeq \operatorname{colim}(C(C \otimes \alpha_x) \twoheadleftarrow C(\alpha_x) \to C(\alpha_y)). \end{aligned}$$

We then have a sequence of complicial exact form categories

$$(\mathcal{E}, Q, w, \mathbb{D}\ \eta) \to (\operatorname{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{pt}}, \mathbb{D}, \eta) \to (\operatorname{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{cone}}, \mathbb{D}, \eta),$$

where the first functor is $x \mapsto \mathrm{id}_x$ with duality compatibility the identity $\mathrm{id}_{\mathbb{D}(x)} = \mathbb{D}(\mathrm{id}_x)$, and the latter is the change of weak equivalences functor, with underlying functor the identity. Recall that the poset $[1]$ is given the unique strict duality $i \mapsto 1 - i$, and that for a map $f : x \to y$ in $\mathcal{E}$, we define $Q_{[1]}(f)$ as the pullback

$$\begin{array}{ccc} Q_{[1]}(f) & \longrightarrow & Q(x) \\ \downarrow & & \downarrow \rho_x \\ \operatorname{Hom}_{\mathcal{E}}(x, \mathbb{D}(y)) & \xrightarrow{\mathbb{D}(f)_*} & \operatorname{Hom}_{\mathcal{E}}(x, \mathbb{D}(x)), \end{array} \tag{5.1}$$

inducing a duality $(x \xrightarrow{f} y) \mapsto (\mathbb{D}(y) \xrightarrow{\mathbb{D}(f)} \mathbb{D}(x))$, which is strong with respect to w_{pt}. Write $\operatorname{Ar}(\mathcal{E})^{w_{\mathrm{cone}}} \subset \operatorname{Ar}(\mathcal{E})$ for the subcategory of w_{cone}-acyclic objects. Then the functor $(\mathcal{E}, Q, w, \mathbb{D}, \eta) \to (\operatorname{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{pt}}, \mathbb{D}_{[1]}, \eta)$ factors through the duality-preserving inclusion

$$(\operatorname{Ar}(\mathcal{E})^{w_{\mathrm{cone}}}, Q_{[1]}, w_{\mathrm{pt}}, \mathbb{D}, \eta) \hookrightarrow (\operatorname{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{pt}}, \mathbb{D}, \eta),$$

and by [Sch24b, Lem. 11.3] there is an induced equivalence

$$|w\mathcal{Q}\mathrm{uad}(S^e_\bullet(\mathcal{E}, Q, w))| \xrightarrow{\simeq} \left|w\mathcal{Q}\mathrm{uad}(S^e_\bullet(\operatorname{Ar}(\mathcal{E})^{w_{\mathrm{cone}}}, Q_{[1]}, w_{\mathrm{pt}}))\right|$$

Write $(\mathcal{E}^{[n]}, Q_{[n]}, w)$ for the complicial exact form category with weak equivalences $(\mathcal{E}^{[n-1]}, Q_{[n-1]}, w)^{[1]}$ for $n \geq 2$, where $(\mathcal{E}, Q, w)^{[1]} = (\operatorname{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{cone}})$. Then by [Sch24b, Prop. 11.4], there are canonical maps for each $n \geq 0$,

$$\left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}^{(n)}_\bullet(\mathcal{E}^{[n]}, Q_{[n]}, w))\right| \to \Omega \left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}^{(n+1)}_\bullet(\mathcal{E}^{[n+1]}, Q_{[n+1]}, w))\right|, \tag{5.2}$$

which for $n \geq 1$ are equivalences, and for $n = 0$ factors as

$$|w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w)| \to \mathcal{GW}(\mathcal{E}, Q, w) \xrightarrow{\simeq} \Omega \left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}^{(1)}_\bullet(\mathcal{E}^{[1]}, Q_{[1]}))\right|.$$

Here we write $\mathcal{R}^{(n)}_\bullet(\mathcal{E}, Q, w)$ for the diagonal of the n^{th} iterate of the subdivided $S_\bullet$-construction on $(\mathcal{E}, Q, w)$, i.e. the multisimplicial exact form category

$$(\Delta^{\mathrm{op}})^{\times n} \to \mathcal{F}\mathrm{orm}\mathcal{C}\mathrm{at}, \quad ([k_1], \dots, [k_n]) \mapsto S^e_{k_1} S^e_{k_2} \dots S^e_{k_n}(\mathcal{E}, Q, w).$$

Definition 5.1.1 ([Sch24b, Def. 11.5]). The Grothendieck-Witt spectrum associated to a complicial exact form category with weak equivalences is the (spectrification of the) positive Ω-spectrum

$$\mathrm{GW}(\mathcal{E}, Q, w) := \left(|w\mathcal{Q}\mathrm{uad}(\mathcal{E}, Q, w)|, \left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}^{(1)}_\bullet(\mathcal{E}^{[1]}, Q_{[1]})\right|, \left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}^{(2)}_\bullet(\mathcal{E}^{[2]}, Q_{[2]})\right|, \dots\right),$$

with bonding maps those given in (5.2). For $n \geq 0$, define the n-shifted Grothendieck-Witt spectrum

$$\mathrm{GW}^{[n]}(\mathcal{E}, Q, w) := \mathrm{GW}(\mathcal{E}^{[n]}, Q_{[n]}, w).$$

5.2 Delooping the hermitian K-theory of Poincaré categories

Recollection 5.2.1. We have the Rezk adjunction

$$s\mathcal{S} \underset{N}{\overset{\mathrm{acat}}{\rightleftarrows}} \mathcal{C}\mathrm{at}_\infty$$

presenting $\mathcal{C}\mathrm{at}_\infty$ of (small) ∞-categories as a reflective subcategory of the ∞-category of (small) simplicial spaces, with the essential image of the inclusion $N = \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\infty}(\Delta^\bullet, -)$ the ∞-category of **complete Segal spaces**: a simplicial space $X_\bullet$ is Segal if the maps $\mathrm{seg}_i : [1] \to [n], 0, 1 \mapsto i-1, i$ induce an equivalence

$$(\mathrm{seg}_1, \ldots, \mathrm{seg}_n) : X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

where the iterated pullback is induced by

$$\begin{array}{ccc} [0] & \xrightarrow{d^1} & [1] \\ \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle \mathrm{seg}_i} \\ [1] & \xrightarrow{\mathrm{seg}_{i-1}} & [n] \end{array}$$

for $1 \le i \le n$. $X_\bullet$ is additionally complete if the square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_3 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle (d_{12}, d_{03})} \\ X_0 \times X_0 & \longrightarrow & X_1 \times X_1, \end{array}$$

with horizontal maps the degeneracies, is cartesian in $\mathcal{S}$. We refer to the left adjoint acat as the **associated category**. Given a Segal space $X_\bullet$ (not necessarily complete) and $x, y \in X_0$, we may recover the mapping space $\mathrm{Map}_{\mathrm{acat}(X)}(x, y)$ as the pullback in spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{acat}(X)}(x, y) & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_1, d_0)} \\ \Delta^0 & \xrightarrow{(x,y)} & X_0 \times X_0, \end{array} \tag{5.3}$$

by [CDH+II, §2.1].

Recall [CDH+II] that a sequence

$$(\mathcal{C}, \Omega) \xrightarrow{(i,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\vartheta)} (\mathcal{E}, \Psi) \tag{5.4}$$

in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{p}}$ is a *Poincaré-Verdier sequence* if it is both a fibre and cofibre sequence; in this case the image of (5.4) under the canonical functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{p}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$ is a Verdier sequence, and we say (5.4) is split if the underlying Verdier sequence splits, i.e. p (equivalently i) admits both a left and right adjoint. In this situation, we refer to (i, η) as a (split) Poincaré-Verdier inclusion, and (p, ϑ) as a (split) Poincaré-Verdier projection. A cartesian square

$$\begin{array}{ccc} (\mathcal{C}, \Phi) & \longrightarrow & (\mathcal{D}, \Psi) \\ \downarrow & & \downarrow \\ (\mathcal{C}', \Phi') & \longrightarrow & (\mathcal{D}', \Psi') \end{array}$$

is then (split) Poincaré-Verdier if each of the vertical functors are (split) Poincaré-Verdier projections.

A functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ is Verdier localising (resp. additive) if it sends Poincaré-Verdier (resp. split Poincaré-Verdier) squares to cartesian squares of spaces, and grouplike if it is additive, and the canonical lift to $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ arising from semi-additivity of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ takes values in the subcategory of grouplike $\mathbb{E}_\infty$-spaces; see [CDH+II, §1.5]. Given an additive functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$, to any Poincaré category $(\mathcal{C}, \Omega)$ we obtain an associated simplicial space $\mathcal{F}\mathcal{Q}_\bullet(\mathcal{C}, \Omega^{[1]})$ which is Segal for $\mathcal{F}$ additive, and additionally complete if $\mathcal{F}$ preserves limits. The split Poincaré-Verdier sequence

$$(\mathcal{C}, \Omega) \to \mathcal{Q}_1(\mathcal{C}, \Omega^{[1]}) \xrightarrow{(d_1, d_0)} \mathcal{Q}_0(\mathcal{C}, \Omega^{[1]})^{\times 2} = (\mathcal{C}, \Omega^{[1]})^{\times 2},$$

where the left-hand map is informally given by $x \mapsto (0 \leftarrow x \to 0)$ with quadratic compatibility the canonical unit equivalence $\Omega(x) \to \Omega^{[1]}_1(0 \leftarrow x \to 0) = \Omega\Sigma\Omega(x)$, yields upon applying $\mathcal{F}$ a cartesian square of spaces

$$\begin{array}{ccc} \mathcal{F}(\mathcal{C}, \Omega) & \longrightarrow & \mathcal{F}\mathcal{Q}_1(\mathcal{C}, \Omega^{[1]}) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(0,0)} & \mathcal{F}(\mathcal{C}, \Omega^{[1]})^{\times 2}. \end{array} \tag{5.5}$$

Writing $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ for the ∞-category associated to the Segal space $\mathcal{F}\mathcal{Q}_\bullet(\mathcal{C}, \Omega^{[1]})$, the square (5.3) gives an identification, for $x, y \in \mathcal{F}(\mathcal{C}, \Omega^{[1]})$,

$$\mathrm{Map}_{\mathrm{Cob}(\mathcal{C},\Omega)}(x, y) \simeq \mathrm{fib}_{(x,y)}\left(\mathcal{F}\mathcal{Q}_1(\mathcal{C}, \Omega^{[1]}) \to \mathcal{F}(\mathcal{C}, \Omega^{[1]})^{\times 2}\right),$$

Accordingly, (5.5) induces a map $\mathcal{F}(\mathcal{C}, \Omega) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Omega)}(0, 0) \to \Omega\left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)\right|$. For additive $\mathcal{F}$, the functor $(\mathcal{C}, \Omega) \mapsto \left|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)\right|$ is again additive, and the assignment $\mathcal{F} \mapsto \left|\mathrm{Cob}^{\mathcal{F}}(-)\right|$ assembles into an endofunctor $\left|\mathrm{Cob}^{(-)}(-)\right|$ of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$. By [CDH+II, Th. 3.3.4], $\left|\mathrm{Cob}^{\mathcal{F}}(-)\right|$ is a model for the suspension of $\mathcal{F}$ in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$, and the map $\mathcal{F} \to \Omega\left|\mathrm{Cob}^{\mathcal{F}}(-)\right|$ exhibits the target as the group completion of $\mathcal{F}$. For $n \geq 0$, write

$$\mathcal{Q}^{(n)}_{\bullet,\dots,\bullet}(\mathcal{C}, \Omega) : (\Delta^{\mathrm{op}})^{\times n} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}, \quad ([k_1], \dots, [k_n]) \mapsto \mathcal{Q}_{k_1}\mathcal{Q}_{k_2}\dots\mathcal{Q}_{k_n}(\mathcal{C}, \Omega)$$

for the n-simplicial hermitian $\mathcal{Q}$-construction on $(\mathcal{C}, \Omega)$, $\mathcal{Q}^{(n)}_\bullet$ for the diagonal, and $\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, \Omega) := \mathcal{F}\mathcal{Q}^{(n)}_\bullet(\mathcal{C}, \Omega^{[n]})$. We have $\mathrm{Cob}^{\mathcal{F}}_1(\mathcal{C}, \Omega) = \mathcal{F}\mathcal{Q}_\bullet(\mathcal{C}, \Omega^{[1]})$, and canonical identifications

$$\left|\mathrm{Cob}_1^{|\mathrm{Cob}^{\mathcal{F}}_n(-)|}(\mathcal{C}, \Omega)\right| \simeq \left|\mathcal{F}\mathcal{Q}^{(n+1)}_\bullet(\mathcal{C}, \Omega^{[n+1]})\right| = |\mathrm{Cob}_{n+1}(\mathcal{C}, \Omega)|.$$

Then for additive $\mathcal{F}$, by [CDH+II, Prop. 3.4.5], the maps

$$\left|\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, \Omega)\right| \to \Omega\left|\mathrm{Cob}^{\mathcal{F}}_{n+1}(\mathcal{C}, \Omega)\right|$$

adjoint to the equivalences $\Sigma|\mathrm{Cob}^{\mathcal{F}}_n(-)| \simeq |\mathrm{Cob}_1^{|\mathrm{Cob}^{\mathcal{F}}_n(-)|}|$ are equivalences for $n \geq 1$, and we write $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(-)$ for the associated positive Ω-spectrum, with $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, \Omega)_n = \left|\mathrm{Cob}^{\mathcal{F}}_n(\mathcal{C}, \Omega)\right|$. For $\mathcal{F}$ grouplike, $\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ is an Ω-spectrum, and the group completion map $\mathcal{F} \to \mathcal{F}^{\mathrm{grp}}$ induces the spectrification map

$$\mathbb{C}\mathrm{ob}^{\mathcal{F}}(\mathcal{C}, \Omega) \to \mathbb{C}\mathrm{ob}^{\mathcal{F}^{\mathrm{grp}}}(\mathcal{C}, \Omega)$$

for each $(\mathcal{C}, \Omega) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$. Recall that $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ is additive, with group completion $\mathcal{GW}$, and write $\mathrm{Cob}_n := \mathrm{Cob}^{\mathrm{Pn}}_n$.

Definition 5.2.2. The Grothendieck-Witt spectrum associated to a Poincaré category $(\mathcal{C}, \Omega)$ is the (spectrification of the) positive Ω-spectrum

$$\mathrm{GW}(\mathcal{C}, \Omega) := (|\mathrm{Cob}_0(\mathcal{C}, \Omega)|, |\mathrm{Cob}_1(\mathcal{C}, \Omega)|, |\mathrm{Cob}_2(\mathcal{C}, \Omega)|, \dots).$$

For $n \in \mathbb{Z}$ we set $\mathrm{GW}^{[n]}(\mathcal{C}, \Omega) := \mathrm{GW}(\mathcal{C}, \Omega^{[n]})$, the n-shifted Grothendieck-Witt (pre-)spectrum. As above, the map $\mathrm{Pn}(\mathcal{C}, \Omega) = |\mathrm{Cob}_0^{\mathrm{Pn}}(\mathcal{C}, \Omega)| \to \Omega |\mathrm{Cob}_1^{\mathrm{Pn}}(\mathcal{C}, \Omega)| \simeq \mathcal{GW}(\mathcal{C}, \Omega)$ is the universal map exhibiting the functor $\mathcal{GW} : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}$ as the group completion of Pn.

5.3 Comparing Grothendieck-Witt spectra

Fix a complicial exact form category with weak equivalences $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$, with derived Poincaré category $(\mathrm{L}_w(\mathcal{E}), \mathbf{R}Q)$. Recall from 5.1 that $\mathrm{Ar}(\mathcal{E})$ carries a pointwise structure of a complicial exact category with weak equivalences, and write $w_{\mathrm{pt}} \subset \mathrm{Ar}(\mathcal{E})$ for the subcategory of pointwise weak equivalences. By [Cis19, Th. 7.6.17], the functor $\mathrm{Ar}(\mathcal{E}) \to \mathrm{Ar}(\mathrm{L}_w(\mathcal{E}))$ induces an equivalence $\mathrm{L}_{w_{\mathrm{pt}}}(\mathrm{Ar}(\mathcal{E})) \xrightarrow{\simeq} \mathrm{Ar}(\mathrm{L}_w(\mathcal{E}))$, and we write $(\mathrm{Ar}(\mathrm{L}_w(\mathcal{E})), \mathbf{R}(Q_{[1]}))$ for the derived Poincaré category associated to the form category $(\mathrm{Ar}(\mathcal{E}), Q_{[1]}, w_{\mathrm{pt}}, \mathbb{D}, \eta)$ of 5.1.

Recollection 5.3.1. Given a Poincaré category $(\mathcal{C}, \Omega)$, the arrow category $\mathrm{Ar}(\mathcal{C})$ can be equipped with a quadratic functor Ω_{ar}, defined pointwise by the cartesian square

$$\begin{array}{ccc} \Omega_{\mathrm{ar}}(x \xrightarrow{f} y) & \longrightarrow & \Omega(x) \\ \downarrow & & \downarrow \\ B_\Omega(y, x) & \longrightarrow & B_\Omega(x, x), \end{array}$$

and by [CDH⁺I, §2.4], the pair $(\mathrm{Ar}(\mathcal{C}), \Omega_{\mathrm{ar}})$ is Poincaré if $(\mathcal{C}, \Omega)$ is.

We may also equip $\mathrm{Ar}(\mathcal{C})$ with the metabolic Poincaré structure $\mathrm{Met}(\mathcal{C}, \Omega)$, with quadratic functor

$$\Omega_{\mathrm{met}}(f) := \mathrm{fib}(\Omega(y) \xrightarrow{\Omega(f)} \Omega(x)),$$

inducing a duality

$$(x \xrightarrow{f} y) \mapsto (\mathrm{fib}(\mathbb{D}(y) \to \mathbb{D}(x)) \to \mathbb{D}(y)).$$

$\mathrm{Met}(\mathcal{C}, \Omega)$ participates in the metabolic (split) Poincaré-Verdier sequence

$$(\mathcal{C}, \Omega) \to \mathrm{Met}(\mathcal{C}, \Omega^{[1]}) \to (\mathcal{C}, \Omega^{[1]}),$$

with the first functor informally given by $x \mapsto (x \to 0)$, and the second by $(x \to y) \mapsto y$, with quadratic compatibilities

$$\Omega(x) \xrightarrow{\simeq} \mathrm{fib}(0 \to \Sigma\Omega(x)) \quad \text{ and } \Sigma\,\mathrm{fib}\left(\Omega(y) \xrightarrow{\Omega(f)} \Omega(x)\right) \to \Sigma\Omega(y).$$

Writing $S_2\mathcal{C} \subset \mathrm{Fun}(\Delta^2, \mathcal{C})$ for the full subcategory on fibre-cofibre sequences in $\mathcal{C}$, by [CDH⁺I, Lem. 2.4 .5] the zig-zag of equivalences

$$\begin{array}{ccccc} \mathrm{Ar}(\mathcal{C}) & \xleftarrow{d_2} & S_2(\mathcal{C}) & \xrightarrow{d_0} & \mathrm{Ar}(\mathcal{C}) \\ (x \to y) & \longleftarrow & (x \to y \to z) & \longmapsto & (y \to z) \end{array}$$

induces an equivalence of quadratic functors

$$d_2^*\mathcal{Q}_{\mathrm{ar}} \simeq d_0^*\mathcal{Q}_{\mathrm{met}}^{[1]}.$$

Remark 5.3.2. By [CDH⁺I, Prop. 7.3.15, 7.3.20, Rem. 7.3.17, 7.3.21], the forgetful functor $U : \mathcal{C}at_\infty^p \to \mathcal{C}at_\infty^h$ admits both adjoints:

$$\mathcal{C}at_\infty^p \underset{\mathrm{Pair}(-,-)}{\overset{U}{\rightleftarrows}} \mathcal{C}at_\infty^h \underset{U}{\overset{\mathrm{Pair}(-,-^{[\sigma]})}{\rightleftarrows}} \mathcal{C}at_\infty^p,$$

and moreover for $(\mathcal{C},\mathcal{Q})$ Poincaré we have natural equivalences $\mathrm{Pair}(\mathcal{C},\mathcal{Q}) \simeq (\mathrm{Ar}(\mathcal{C}),\mathcal{Q}_{\mathrm{ar}})$, and $\mathrm{Pair}(\mathcal{C},\mathcal{Q}^{[\sigma]}) \simeq \mathrm{Met}(\mathcal{C},\mathcal{Q})$, for $\mathcal{Q}^{[\sigma]}(x) := \Omega\mathcal{Q}(\Omega x)$. Accordingly, both limits and colimits in $\mathcal{C}at_\infty^p$ are computed on underlying hermitian categories, and the functors $\mathrm{Met}(-,-)$ and $\mathrm{Ar}(-,-)$ each preserve limits and colimits when considered as endofunctors on $\mathcal{C}at_\infty^p$: given a diagram $(\mathcal{C}_\bullet,\mathcal{Q}_\bullet) : I \to \mathcal{C}at_\infty^p$, we have

$$\begin{aligned}\operatorname{colim}_I^p \mathrm{Met}(\mathcal{C}_i,\mathcal{Q}_i) \simeq \operatorname{colim}_I^p \mathrm{Pair}(\mathcal{C}_i,\mathcal{Q}_i^{[\sigma]}) &\simeq \mathrm{Pair}(-,-^{[\sigma]})\left(\operatorname{colim}_I^h(\mathcal{C}_i,\mathcal{Q}_i)\right)\\ &\simeq \mathrm{Pair}(-,-^{[\sigma]})\left(\operatorname{colim}_I^p(\mathcal{C}_i,\mathcal{Q}_i)\right) \simeq \mathrm{Met}(\operatorname{colim}_I^p(\mathcal{C}_i,\mathcal{Q}_i)),\end{aligned}$$

where the decoration on the colim indicates the ambient category, and dually for $\mathrm{Ar}(-,-)$ and limits. The dual statements follow from the equivalence $(\mathrm{Ar}(\mathcal{C}),\mathcal{Q}_{\mathrm{ar}}) \simeq \mathrm{Met}(\mathcal{C},\mathcal{Q}^{[1]}) \simeq \mathrm{Met}(\mathcal{C},\mathcal{Q})^{[1]}$ and the fact that the shift functor $(\mathcal{C},\mathcal{Q}) \mapsto (\mathcal{C},\mathcal{Q}^{[n]})$ preserves limits and colimits for each $n \in \mathbb{Z}$.

Lemma 5.3.3. *There is a natural Poincaré equivalence*

$$(L_{w_{\mathrm{pt}}}(\mathrm{Ar}(\mathcal{E})),\mathbf{R}(Q_{[1]})) \simeq (\mathrm{Ar}(L_w(\mathcal{E})),(\mathbf{R}Q)_{\mathrm{ar}}).$$

Proof. Assume for now that $w = w_{\mathrm{Frob}}$ is the class of Frobenius equivalences in $\mathcal{E}$, and that $\mathcal{E} = \mathcal{E}_{\mathrm{Frob}}$ is equipped with the Frobenius exact structure. Then by the results of §3, there are equivalences

$$\Omega^\infty\mathbf{R}_{\mathrm{Frob}}Q \simeq Q^{\Delta^\bullet}, \quad \Omega^\infty\mathbf{R}_{\mathrm{Frob}}(Q_{[1]}) \simeq (Q_{[1]})^{\Delta^\bullet},$$

where we write $\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})$ for the right derived quadratic functor of $Q_{[1]}$ at the Frobenius equivalences internal to $\mathrm{Ar}(\mathcal{E})$. We claim that the Dwyer-Kan localisation $\gamma : L_{\mathrm{Frob}}(\mathrm{Ar}(\mathcal{E})) \to L_{\mathrm{pt}}(\mathrm{Ar}(\mathcal{E})) \simeq \mathrm{Ar}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))$ at the pointwise Frobenius equivalences is an equivalence. For this, it suffices to show that the classes of pointwise and internal Frobenius equivalences have equivalent subcategories of acyclics (a map in a complicial exact category with weak equivalences $(\mathcal{E},w)$ is in w precisely when its cone is acyclic, see Remark A.3.10). Given a pointwise Frobenius contractible map $f : x \to y$, we wish to show there is some Frobenius contractible object $f' : x' \to y'$ of $\mathrm{Ar}(\mathcal{E}_{\mathrm{Frob}})$ with f a retract in $\mathrm{Ar}(\mathcal{E}_{\mathrm{Frob}})$ of f'. Since each map factors as a composition $x \rightarrowtail x \oplus Py \twoheadrightarrow y$ of a Frobenius ingression followed by a Frobenius egression, both of which in this case are Frobenius equivalences, and (internally) Frobenius contractible maps are closed under composition, we may suppose that f is a Frobenius ingression (the case for egressions is dual). We then note that f admits a retraction, as a Frobenius ingression between Frobenius injectives, so that since it admits a cokernel it is equivalent to a canonical inclusion $x \to x \oplus x'$. Now since $x \oplus x'$ is Frobenius acyclic by assumption, the map $i_{x\oplus x'} : x\oplus x' \to C\otimes(x\oplus x') \cong (C\otimes x)\oplus(C\otimes x')$ admits a retraction r, inducing retractions r_x and $r_{x'}$ of $x \rightarrowtail C\otimes x$ and $x' \rightarrowtail C\otimes x'$ via pre- and post-composition with the canonical inclusions and projections. We may replace r with the sum $r_x \oplus r_{x'}$, and obtain a diagram

$$\begin{array}{ccccc} x & \xrightarrow{\quad i_x \quad} & C\otimes x & \xrightarrow{\quad r_x \quad} & x \\ \downarrow{\scriptstyle \binom{\mathrm{id}_x}{0}} & & \downarrow{\scriptstyle \binom{\mathrm{id}_{C\otimes x}}{0}} & & \downarrow{\scriptstyle \binom{\mathrm{id}_x}{0}} \\ x\oplus x' & \xrightarrow{i_x\oplus i_{x'}} & (C\otimes x)\oplus(C\otimes x') & \xrightarrow{r_x\oplus r_{x'}} & x\oplus x', \end{array}$$

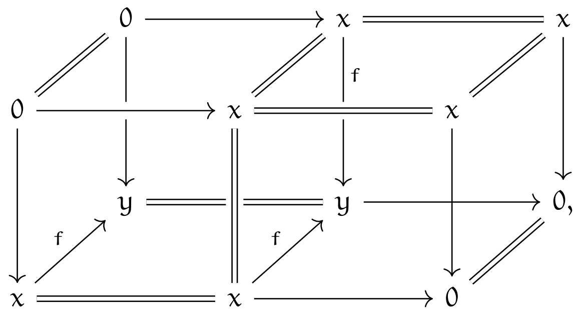
so that the Frobenius ingression $(f : x \rightarrowtail y) \rightarrowtail (C \otimes f : C \otimes x \rightarrowtail C \otimes y)$ internal to $\mathrm{Ar}(\mathcal{E}_{\mathrm{Frob}})$ admits a retraction, as required. Note accordingly that

$$\Omega^\infty \mathbf{R}_{\mathrm{Frob}_{\mathrm{pt}}}(Q_{[1]})(\mathrm{id}_x) = \Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(\mathrm{id}_x) = Q_{[1]}(\mathrm{id}_{\Delta^\bullet x}) = Q^{\Delta^\bullet}(x),$$

and

$$\Omega^\infty \mathbf{R}_{\mathrm{Frob}_{\mathrm{pt}}}(Q_{[1]})(0 \to x) = Q_{[1]}(0 \to \Delta^\bullet x) = 0,$$

where the last line follows from the definition of $Q_{[1]}$ (5.1). Now for $f \in \mathrm{Ar}(\mathcal{E})$, consider the diagram in $\mathrm{Ar}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))$



furnished by f, considered as a map of fibre sequences from the front to the back composite faces (that these are fibre sequences follows since they are pointwise Frobenius congressions, and so taken to exact sequences by the localisation at the pointwise Frobenius equivalences). Since $\Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})$ is 2-excisive, there is an induced equivalence of total fibres

$$\mathrm{fibt}\left[\begin{array}{ccc} \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(f) & \longrightarrow & \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(0 \to y) \\ \downarrow & & \downarrow \\ B_{\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})}(f, 0 \to y) & \longrightarrow & B_{\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})}(0 \to y, 0 \to y) \end{array}\right] \xrightarrow{\simeq} \mathrm{fibt}\left[\begin{array}{ccc} \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(\mathrm{id}_x) & \longrightarrow & \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(0 \to x) \\ \downarrow & & \downarrow \\ B_{\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})}(\mathrm{id}_x, 0 \to x) & \longrightarrow & B_{\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})}(0 \to x, 0 \to x), \end{array}\right]$$

and upon passing to connective covers and commuting fibres, we obtain a cartesian square

$$\begin{array}{ccc} \Omega^\infty \mathbf{R}_{\mathrm{Frob}}(Q_{[1]})(f) & \longrightarrow & \Omega^\infty \mathbf{R}_{\mathrm{Frob}} Q(x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_\Delta(y, \mathbb{D}(x)) & \longrightarrow & \mathrm{Map}_\Delta(x, \mathbb{D}(x)), \end{array}$$

and a natural equivalence $\Omega^\infty \mathbf{R}(Q_{[1]})(f) \simeq \Omega^\infty (\mathbf{R}Q)_{\mathrm{ar}}(f)$, which by Proposition 3.2.6 promotes to an equivalence of $\mathcal{S}\mathrm{p}$-valued quadratic functors $\mathbf{R}(Q_{[1]}) \simeq (\mathbf{R}Q)_{\mathrm{ar}}$ on $\mathrm{Ar}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))$. Here we have used that $B_{\mathbf{R}_{\mathrm{Frob}}(Q_{[1]})}(0 \to x, 0 \to x) \simeq \mathrm{Map}_{\mathrm{Ar}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))}(0 \to x, \mathbb{D}(0 \to x))$ can be computed as the mapping space in the complicial exact category $\mathrm{Ar}(\mathcal{E}_{\mathrm{Frob}})$, and is contractible, and similarly

$$B_{\mathbf{R}_{\mathrm{Frob}} Q_{[1]}}(f, 0 \to y) \simeq \mathrm{Map}_{\mathrm{Ar}(L_{\mathrm{Frob}}(\mathcal{E}_{\mathrm{Frob}}))}(f, \mathbb{D}(0 \to y)) \simeq \mathrm{Map}_\Delta(x, \mathbb{D}(y)).$$

To conclude for general $w \subset \mathcal{E}$, we simply note that the functor $(\mathcal{C}, \Omega) \mapsto (\mathrm{Ar}(\mathcal{C}), \Omega_{\mathrm{ar}})$ commutes with cofibres in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{p}}$ by Remark 5.3.2, so the derived functor $\mathbf{R}(Q_{[1]})$ of $Q_{[1]}$ at the pointwise (w-)weak equivalences is naturally equivalent to $(\mathbf{R}Q)_{\mathrm{ar}}$ on $\mathrm{Ar}(L_w(\mathcal{E})) \simeq L_{w_{\mathrm{pt}}}(\mathrm{Ar}(\mathcal{E}))$. □

We thus have an equivalence of Poincaré-Verdier inclusions

$$\begin{array}{ccc} (L_w(\mathcal{E}), \mathbf{R}Q) & \hookrightarrow & (\mathrm{Ar}(L_w(\mathcal{E})), (\mathbf{R}Q)_{\mathrm{ar}}) \\ \| & & \downarrow \wr \\ (L_w(\mathcal{E}), \mathbf{R}Q) & \hookrightarrow & \mathrm{Met}(L_w(\mathcal{E}), \mathbf{R}Q^{[1]}), \end{array} \qquad (5.6)$$

with the right vertical arrow informally given by $(x \xrightarrow{f} y) \mapsto (y \to \mathrm{cofib}(f))$. The functor $L_w(\mathcal{E}) \to \mathrm{Ar}(L_w(\mathcal{E}))$, $x \mapsto \mathrm{id}_x$ clearly lands in the stable subcategory spanned by the cone-acyclic maps, and participates in the adjoint pair

$$L_w(\mathcal{E}) \underset{s}{\overset{x\mapsto \mathrm{id}_x}{\underset{\perp}{\rightleftarrows}}} \mathrm{Ar}(L_w(\mathcal{E}))$$

where s is the source functor. Since $\mathrm{Ar}(L_w(\mathcal{E}))^{w_{\mathrm{cone}}}$ is simply the full subcategory spanned by the equivalences in $L_w(\mathcal{E})$, this restricts to an equivalence

$$L_w(\mathcal{E}) \simeq \mathrm{Ar}(L_w(\mathcal{E}))^{w_{\mathrm{cone}}},$$

and accordingly the upper Poincaré-Verdier inclusion of (5.6) is equivalent to

$$(\mathrm{Ar}(L_w(\mathcal{E})^{w_{\mathrm{cone}}}), (\mathbf{R}Q)_{\mathrm{ar}}\,|_{\mathrm{Ar}(L_w(\mathcal{E})^{w_{\mathrm{cone}}})}) \hookrightarrow (\mathrm{Ar}(L_w(\mathcal{E}), (\mathbf{R}Q)_{\mathrm{ar}}).$$

Writing $\pi : \mathrm{Ar}(L_w(\mathcal{E})) \to L_{\mathrm{cone}}(\mathrm{Ar}(\mathcal{E}))$ for the localisation at w_{cone}, we obtain:

Corollary 5.3.4. *There is a natural Poincaré equivalence*

$$\left(L_{\mathrm{cone}}(\mathrm{Ar}(\mathcal{E})), \pi_!(\mathbf{R}(Q_{[1]}))\right) \simeq \left(L_w(\mathcal{E}), (\mathbf{R}Q)^{[1]}\right).$$

Corollary 5.3.5. *Let* $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ *be a complicial exact form category with weak equivalences, with derived Poincaré category* $(L_w(\mathcal{E}), \Omega)$*. Then for each* $n \geq 0$*, there is a natural equivalence of spaces*

$$\left|\mathrm{wQuad}\,\mathcal{R}_\bullet^{(n)}(\mathcal{E}, Q, w)^{[n]}\right| \xrightarrow{\simeq} |\mathrm{Cob}_n(L_w(\mathcal{E}, \mathbf{R}Q))|$$

Proof. We show inductively that the complicial exact form category $(\mathcal{E}, Q, w)^{[n]}$ has derived Poincaré category $(L_w(\mathcal{E}), \mathbf{R}Q^{[n]})$: the case $n = 0$ holds by definition. By Corollary 5.3.4, for $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ any complicial exact form category with weak equivalences with derived Poincaré category $(L_w(\mathcal{E}), \mathbf{R}Q)$, the derived Poincaré category of $(\mathcal{E}, Q, w)^{[1]}$ is (Poincaré-equivalent to) $\left(L_w(\mathcal{E}), (\mathbf{R}Q)^{[1]}\right)$. Since for any $n \geq 1$ and any indices $k_1, \ldots, k_n \in \mathbb{N}$, $\mathcal{R}^{(n)}_{k_1,\ldots,k_n}(\mathcal{E}, Q, w)$ is levelwise a complicial exact form category with weak equivalences, there are natural equivalences

$$\begin{aligned}\left|\mathrm{wQuad}\,\mathcal{R}_\bullet^{(n)}(\mathcal{E}^{[n]}, Q_{[n]}, w^{[n]})\right| &\simeq \left|\mathrm{wQuad}\,\mathcal{R}_\bullet^{(n)}(\mathcal{E}^{[n-1]}, Q_{[n-1]}, w^{[n-1]})^{[1]}\right| \\ &\simeq \left|\mathrm{Pn}\mathcal{Q}_\bullet^{(n)}\left(L_{w^{[n-1]}}(\mathcal{E}^{[n-1]}), (\mathbf{R}Q_{n-1})^{[1]}\right)\right| \\ &\simeq \left|\mathrm{Pn}\mathcal{Q}_\bullet^{(n)}(L_w(\mathcal{E}), (\mathbf{R}Q)^{[n]})\right| = |\mathrm{Cob}_n(L_w(\mathcal{E}), \mathbf{R}Q)|,\end{aligned}$$

where the second equivalence uses the natural identification

$$|\,\mathrm{wQuad}\,\mathcal{R}_\bullet(\mathcal{E}, Q, w)| \simeq |\mathrm{PnS}_\bullet^e(L_w(\mathcal{E}), \mathbf{R}Q)| \simeq |\mathrm{Pn}\mathcal{Q}_\bullet(L_w(\mathcal{E}), \mathbf{R}Q)|$$

for $(\mathcal{E}, Q, w)$ any complicial exact form category with weak equivalences. □

For $n \geq 0$, write $\gamma_n : \left|\mathrm{wQuad}\,\mathcal{R}_\bullet^{(n)}(\mathcal{E}, Q, w)^{[n]}\right| \to \Omega \left|\mathrm{wQuad}\,\mathcal{R}_\bullet^{(n+1)}(\mathcal{E}, Q, w)^{[n+1]}\right|$ for the bonding maps constructed in [Sch24b], and $\delta_n : |\mathrm{Cob}_n(L_w(\mathcal{E}), \mathbf{R}Q)| \to \Omega\,|\mathrm{Cob}_{n+1}(L_w(\mathcal{E}), \mathbf{R}Q)|$ for those of [CDH⁺II]. We make the following recollection: given a Poincaré category $(\mathcal{C}, \Omega)$, the shifted metabolic sequence

$$(\mathcal{C}, \Omega^{[-1]}) \to \mathrm{Met}(\mathcal{C}, \Omega) \to (\mathcal{C}, \Omega)$$

is split Poincaré-Verdier, and is hence taken by the additive functor $|\mathrm{Cob}(-)|$ to a fibre sequence of spaces

$$\left|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})\right| \to |\mathrm{Cob}(\mathrm{Met}(\mathcal{C},\Qoppa))| \to |\mathrm{Cob}(\mathcal{C},\Qoppa)|\,.$$

By [CDH$^+$II, Prop. 3.1.11], the boundary map $\Omega\,|\mathrm{Cob}(\mathcal{C},\Qoppa)| \xrightarrow{\partial} \left|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})\right|$ fits into the diagram

$$\begin{array}{ccc} & \mathrm{Pn}(\mathcal{C},\Qoppa) & \\ {}^{\delta_0}\swarrow & & \searrow \\ \Omega\,|\mathrm{Cob}(\mathcal{C},\Qoppa)| & \xrightarrow{\quad\partial\quad} & \left|\mathrm{Cob}(\mathcal{C},\Qoppa^{[-1]})\right|, \end{array} \tag{5.7}$$

with the left-hand map the bonding map induced by the identification $\mathrm{Pn}(\mathcal{C},\Qoppa) \simeq \mathrm{Hom}_{\mathrm{Cob}(\mathcal{C},\Qoppa)}(0,0)$, and the right-hand by the inclusion of 0-simplices $(\mathcal{C},\Qoppa) = \mathcal{Q}_0(\mathcal{C},\Qoppa) \to \mathcal{Q}_\bullet(\mathcal{C},\Qoppa)$. Now, recall from [Sch24b, §11] that associated to a complicial exact form category with weak equivalences $(\mathcal{E},Q,w,\mathbb{D},\eta)$ is a commutative diagram

$$\begin{array}{ccccc} w\mathcal{Q}\mathrm{uad}(\mathcal{E},Q,w) & \longrightarrow & & \longrightarrow & 0 \\ \downarrow{\scriptstyle\iota} & & & & \downarrow \\ w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathcal{E},Q,w) & \xhookrightarrow{\quad I\quad} & w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathrm{Ar}(\mathcal{E}),Q_{[1]},w_{\mathrm{pt}}) & \longrightarrow & wS_\bullet\mathcal{E} \\ \downarrow & & \downarrow & & \\ w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathrm{Ar}(\mathcal{E})^{w_{\mathrm{cone}}},Q_{[1]},w_{\mathrm{cone}}) & \hookrightarrow & w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathrm{Ar}(\mathcal{E}),Q_{[1]},w_{\mathrm{cone}}), & & \end{array}$$

where the functor ι is the inclusion of 0-simplices, and I is levelwise induced by the form functor

$$(\mathcal{E},Q,w) \to (\mathrm{Ar}(\mathcal{E}),Q_{[1]},w), \qquad x \mapsto \mathrm{id}_x\,.$$

That the map

$$\left|w\mathcal{Q}\mathrm{uad}(\mathcal{R}_\bullet(\mathrm{Ar}(\mathcal{E}),Q_{[1]},w_{\mathrm{pt}}))\right| \to |wS_\bullet\mathcal{E}| \tag{5.8}$$

is a weak equivalence follows ultimately for additivity for hermitian K-theory of exact form categories (with weak equivalences) [Sch24b, Th. 3.1 (8.1)], and moreover, the maps in these theorems are natural in the input form category, as seen from inspection of the proof of [Sch24b, Th. 3.1]. The map (5.8) thus admits a functorial homotopy inverse, and accordingly the rectangle

$$\begin{array}{ccc} w\mathcal{Q}\mathrm{uad}(\mathcal{E},Q,w) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\iota} & & \downarrow \\ w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathcal{E},Q,w) & \hookrightarrow & w\mathcal{Q}\mathrm{uad}\,\mathcal{R}_\bullet(\mathrm{Ar}(\mathcal{E}),Q_{[1]},w_{\mathrm{pt}}) \end{array}$$

commutes up to natural homotopy. Taking derived Poincaré categories of each of the above, using Lemma 5.3.3 and Propositions 4.1.3 and 4.2.7, and pasting with the nullcomposite sequence obtained from applying $\mathrm{Pn}(-)$ to the metabolic Poincaré-Verdier sequence, we obtain a diagram

$$\begin{array}{ccc} \mathrm{Pn}(\mathrm{L}_w(\mathcal{E}),\mathbf{R}Q) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathrm{Pn}\mathcal{Q}_\bullet(\mathrm{L}_w(\mathcal{E}),\mathbf{R}Q) & \longrightarrow & \mathrm{Pn}\mathcal{Q}_\bullet(\mathrm{Ar}(\mathrm{L}_w(\mathcal{E})),(\mathbf{R}Q)_{\mathrm{ar}}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Pn}\mathcal{Q}_\bullet(\mathrm{L}_w(\mathcal{E}),(\mathbf{R}Q)^{[1]}) \end{array}$$

of simplicial spaces commuting up to canonical homotopy, which we further identify via (5.6) with

$$\begin{array}{ccc}
\mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathrm{Pn}\mathcal{Q}_\bullet(L_w(\mathcal{E}), \mathbf{R}Q) & \longrightarrow & \mathrm{Pn}\mathcal{Q}_\bullet\mathrm{Met}(L_w(\mathcal{E}), \mathbf{R}Q)^{[1]} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Pn}\mathcal{Q}_\bullet(L_w(\mathcal{E}), (\mathbf{R}Q)^{[1]}),
\end{array}$$

where the lower square encodes the fibre sequence associated to the metabolic Poincaré-Verdier sequence. Taking realisations, we obtain an essentially unique map of spaces $\gamma_0 : \mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q) \to \Omega\,|\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q)|$ rendering the diagram

$$\begin{array}{cccc}
\mathrm{Pn}(L_w(\mathcal{E}), \mathbf{R}Q) & \xrightarrow{\gamma_0} & & \\
 & \Omega\,|\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q)| & \longrightarrow & 0 \\
 & \downarrow \partial & & \downarrow \\
 & \left|\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q^{[-1]})\right| & \longrightarrow & |\mathrm{Cob}(\mathrm{Met}(L_w(\mathcal{E}), \mathbf{R}Q)| \\
 & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & |\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q)|
\end{array}$$

commutative, in which the top square encodes the rotated fibre sequence $\Omega\left|\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q^{[-1]})\right| \to |\mathrm{Cob}(\mathrm{Met}(L_w(\mathcal{E}), \mathbf{R}Q)| \to |\mathrm{Cob}(L_w(\mathcal{E}), \mathbf{R}Q)|$, and the lower diagonal map is the inclusion of 0-simplices. The map γ_0 is determined up to contractible choice by the universal property of the limit, and accordingly is homotopic via some η_0 to δ_0. The same argument applies replacing $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$ with the multisimplicial complicial exact form category with weak equivalences $\mathcal{R}_\bullet^{(n)}(\mathcal{E}, Q, w)^{[n]}$, and accordingly we obtain homotopies η_n between the bonding maps γ_n and δ_n for each $n \geq 0$. Using Proposition 5.3.5, we obtain:

Theorem 5.3.6. *Suppose given a complicial exact form category with weak equivalences* $(\mathcal{E}, Q, w, \mathbb{D}, \eta)$*, with derived Poincaré category* $(L_w(\mathcal{E}), \mathbf{R}Q)$*. Then the localisation* $\mathcal{E} \to L_w(\mathcal{E})$ *induces an equivalence of Grothendieck-Witt spectra*

$$\mathrm{GW}(\mathcal{E}, Q, w) \xrightarrow{\simeq} \mathrm{GW}(L_w(\mathcal{E}), \mathbf{R}Q).$$

6 Genuine symmetric Poincaré structures

In this section we consider the formalism of genuine Poincaré structures of [CDH+I] and [CHN25], showing in cases of interest that these are precisely the derived Poincaré categories of exact form categories.

6.1 Bounded derived Poincaré categories

Given an exact category with duality $(\mathcal{E}, \mathbb{D}, \eta)$ embedding exactly into the complicial exact category with weak equivalences and duality $(\mathrm{Ch}_b(\mathcal{E}), \mathbf{qis}, \mathbb{D}, \eta)$, we have an associated duality-preserving Verdier-projection

$$(\mathrm{K}_b(\mathcal{E}), \mathbb{D}, \eta) \to (\mathrm{D}_b(\mathcal{E}), \mathbb{D}', \eta'),$$

where $\mathbb{D}'$ is the left Kan extension of the composite $\pi \circ \mathbb{D}$ along the localisation $\pi^{\mathrm{op}} : \mathrm{K}_b(\mathcal{E})^{\mathrm{op}} \to \mathrm{D}_b(\mathcal{E})^{\mathrm{op}}$, which exists since $\mathbb{D} : \mathrm{Ch}_b(\mathcal{E})^{\mathrm{op}} \to \mathrm{Ch}_b(\mathcal{E})$ preserves acyclic complexes. Writing $(\mathrm{K}_b(\mathcal{E}), \Omega^s_\oplus)$ and $(\mathrm{D}_b(\mathcal{E}), \Omega^s)$ for the corresponding symmetric Poincaré-structures, one may ask whether in general this promotes to a Poincaré-Verdier projection

$$(\mathrm{K}_b(\mathcal{E}), \Omega^s_\oplus) \to (\mathrm{D}_b(\mathcal{E}), \Omega^s),$$

or equivalently [CDH+II, Cor. 1.1.6] if the induced map $\Omega^s_\oplus \to \pi^*\Omega^s$ exhibits Ω^s as the left Kan extension of $\Omega^s_\oplus$ along π^{op}. This will not be the case for a general duality-preserving Verdier quotient of stable ∞-categories, since a priori there is no reason for the commutation of the filtered colimit computing the mapping spectra $\mathrm{map}_{\mathcal{D}}(-,-)$ and the formation of C_2-homotopy fixed points. The situation at hand is however sufficiently finitary to permit this.

Remark 6.1.1. (i) The below essentially amounts to showing that $\mathrm{K}_b(\mathcal{E}) \to \mathrm{D}_b(\mathcal{E})$ is a bounded Karoubi projection of $\mathrm{Perf}(\mathbb{Z})$-linear stable ∞-categories in the sense of [CHN25, §4]. Since we only need that mapping spectra in $\mathrm{K}_b(\mathcal{E})$ can be computed as mapping complexes in the dg-category $\mathrm{Ch}_b(\mathcal{E})$, we dispense with the full $\mathrm{Perf}(\mathbb{Z})$-linearity; it is the case however that any complicial exact category localises to a $\mathbb{Z}$-linear stable ∞-category, giving us an honest tensoring over $\mathrm{Perf}(\mathbb{Z})$ (see Lemma A.3.15).

(ii) For the remainder of the section we assume $\mathcal{E}$ is **weakly idempotent complete** (i.e., every retract admits a kernel). Since by [Sch24b, Lem. 10.6] for any exact form category with duality $(\mathcal{E}, Q, \mathbb{D}, \eta)$ the weak idempotent completion $\mathcal{E} \to \mathcal{E}^\flat$ enhances to an exact form functor

$$(\mathcal{E}, Q, \mathbb{D}, \eta) \to (\mathcal{E}^\flat, Q^\flat, \mathbb{D}^\flat, \eta^\flat)$$

of exact form categories with duality, such that the induced map on Grothendieck-Witt spaces is an equivalence, this constitutes no real loss of generality. The category $\mathrm{Ch}_b(\mathcal{E}^\flat)$ is also weakly idempotent complete, and the induced functor $\mathrm{Ch}_b(\mathcal{E}) \to \mathrm{Ch}_b(\mathcal{E}^\flat)$ promotes to an exact functor of complicial exact categories with weak equivalences and duality.

Lemma 6.1.2. *For $\mathcal{E}$ weakly idempotent complete, the functor $(\mathrm{K}_b(\mathcal{E}), Ϙ^s_\oplus) \to (\mathrm{D}_b(\mathcal{E}), Ϙ^s)$ is a Poincaré-Verdier projection.*

Proof. Write $\gamma : \mathrm{Ch}_b(\mathcal{E}) \to \mathrm{K}_b(\mathcal{E})$ for the Dwyer-Kan localisation at the Frobenius (chain homotopy) equivalences, and $\pi : \mathrm{K}_b(\mathcal{E}) \to \mathrm{D}_b(\mathcal{E})$ for the Verdier projection. The pointwise formula for left Kan extensions gives for $x \in \mathrm{K}_b(\mathcal{E})$ an identification

$$\pi_!(Ϙ^s_\oplus)(\pi(x)) \simeq \operatorname{colim}\left(\mathcal{J}_x^{op} \to \mathrm{K}_b(\mathcal{E}) \xrightarrow{Ϙ^s_\oplus} \mathrm{D}(\mathbb{Z})\right) = \operatorname*{colim}_{y \in \mathcal{J}_x^{op}} \hom_{\mathrm{K}_b(\mathcal{E})}(y, \mathbb{D}(y))^{hC_2},$$

for $\mathcal{J}_x \subset \mathrm{K}_b(\mathcal{E})_{/x})$ the full subcategory of quasi-isomorphisms over x. Now by [HA, Prop. 1.3.5.21], the standard t-structure on $\mathrm{D}(\mathbb{Z})$ is right-separated with $\mathrm{D}_{\leq 0}(\mathbb{Z})$ stable under filtered colimits, so by [CHN25, Cor. 4.2.10] we see that filtered colimits of uniformly bounded above diagrams in $\mathrm{D}(\mathbb{Z})$ commute with finite type limits, i.e. limits indexed by simplicial sets K with finitely many non-degenerate simplices in each degree. We observe that the duality $\mathbb{D} : \mathrm{Ch}_b(\mathcal{E})^{op} \to \mathrm{Ch}_b(\mathcal{E})$ restricts to an equivalence $\mathrm{Ch}_{[a,b]}(\mathcal{E})^{op} \simeq \mathrm{Ch}_{[-b,-a]}(\mathcal{E})$ for each $b \geq a$, and accordingly for $y \in \mathrm{Ch}_{[a,b]}(\mathcal{E})$, the mapping complex $\underline{\mathrm{Hom}}(y, \mathbb{D}(y))$ is concentrated in degrees $[-2b, -2a]$. The mapping spectrum $\mathrm{map}_{\mathrm{K}_b(\mathcal{E})}(\gamma(y), \mathbb{D}(\gamma(y)))$ hence lies in $\mathrm{D}_{[-2b,-2a]}(\mathbb{Z}) = \mathrm{D}_{\geq -2b}(\mathbb{Z}) \cap \mathrm{D}_{\leq -2a}(\mathbb{Z})$, with respect to the standard t-structure on $\mathrm{D}(\mathbb{Z})$. Given a family of complexes $\{y_j\}_{j \in \mathcal{J}} \subset \mathrm{K}_b(\mathcal{E})$, uniformly bounding the coconnectivity of the family $\{\hom_{\mathrm{K}_b(\mathcal{E})}(y_j, \mathbb{D}(y_j))\}_{j \in \mathcal{J}}$ then amounts to uniformly bounding the connectivity of the y_j.

Suppose $x \in \mathrm{Ch}_{b,\geq 0}(\mathcal{E})$; the general case follows by shifting. Then a quasi-isomorphism over $\gamma(x)$ in $\mathrm{K}_b(\mathcal{E})$ is represented by some map $f : y \to x$ in $\mathrm{Ch}_b(\mathcal{E})$ with acyclic cone; since $\mathcal{E}$ is weakly idempotent complete, any acyclic complex is strictly acyclic [Büh10, Prop. 10.14], and accordingly $\mathrm{cone}(f)$ admits factorisations

$$\cdots \longrightarrow x_1 \oplus y_0 \xrightarrow{\begin{pmatrix} d_1 & -f_0 \\ 0 & -d_0 \end{pmatrix}} x_0 \oplus y_{-1} \xrightarrow{(0\ \ -d_{-1})} y_{-2} \longrightarrow \cdots$$

$$x_1 \oplus y_0 \xrightarrow{(p_0\ \ q_0)} w_0 \xrightarrow{\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}} x_0 \oplus y_{-1}, \qquad x_0 \oplus y_{-1} \xrightarrow{(p_{-1}\ \ q_{-1})} w_{-1} \xrightarrow{j_{-1}} y_{-2}$$

with the sequence $w_0 \rightarrowtail x_0 \oplus y_{-1} \twoheadrightarrow w_{-1}$ a congression in $\mathcal{E}$. Since j_{-1} is monic, the map p_{-1} is necessarily zero, so that $y_{-1} \to w_{-1}$ is egressive. Writing $y_{\geq -2}$ for the complex $\cdots \to y_0 \to y_{-1} \overset{q_{-1}}{\twoheadrightarrow} w_{-1} \to 0 \to \ldots$, the composite map $y_{\geq -2} \rightarrowtail y \to x$ has acyclic cone given by

$$\cdots \to x_1 \oplus y_0 \to x_0 \oplus y_{-1} \twoheadrightarrow w_{-1} \to 0 \ldots$$

(we note that this truncation is by no means sharp). Write $j_x : \mathcal{J}_x^{\geq -2} \hookrightarrow \mathcal{J}_x$ for the full subcategory on quasi-isomorphisms $y \to x$ in $\mathrm{K}_b(\mathcal{E})$ with y concentrated in degrees ≥ -2. It follows from above that the opposite inclusion j_x^{op} is cofinal: since $\mathrm{K}_b(\mathcal{E})$ admits finite limits and colimits, that the same is true for the iterated comma categories $(j_x^{op})_{(y,f)/} = ((j_x)_{/(y,f)})^{op}$ follows from the stability of the class of quasi-isomorphisms under pushout and pullbacks, and these slices are contractible as soon as they are nonemptym. We may thus resolve $\gamma(x) \in \mathrm{K}_b(\mathcal{E})$ by the uniformly bounded below family of complexes indexed by $\mathcal{J}_x^{\geq -2}$, and we are done. □

6.2 Orientations on the derived category

Suppose given an exact category with duality $(\mathcal{E}, \mathbb{D}, \eta)$, and write $(\mathcal{E}_\oplus, \mathbb{D}, \eta)$ for the underlying additive category with duality. The exact duality-preserving functor $(\mathcal{E}_\oplus, \mathbb{D}, \eta) \to (\mathcal{E}, \mathbb{D}, \eta)$ induces a duality-preserving functor of stable ∞-categories with perfect duality $(D_b(\mathcal{E}_\oplus), \mathbb{D}) = (K_b(\mathcal{E}), \mathbb{D}) \to (D_b(\mathcal{E}), \mathbb{D})$. For $\mathcal{C}$ a stable ∞-category with category of Ind-objects $\mathrm{Ind}(\mathcal{C})$, we refer to a t-structure on $\mathrm{Ind}(\mathcal{C})$ as an *orientation*, following [CHN25].

For $x \in \mathcal{E}$, write P_x for the set of egressions over x. Then the families $(P_x)_{x\in\mathcal{E}}$ generate a Grothendieck topology on x, for which an additive presheaf of abelian groups $\mathcal{E}^{op} \to \mathcal{A}b$ is an additive sheaf precisely if it sends congressions in $\mathcal{E}$ to left-exact sequences of abelian groups [Büh10, App. A]. Write $\mathcal{E}^{lex} := \mathrm{Sh}^{\mathcal{A}b}(\mathcal{E})$ for the Gabriel-Quillen abelian hull of such sheaves, a Grothendieck abelian category with system of compact generators given by the image under the Yoneda embedding of $\mathcal{E}$; in the case $\mathcal{E}$ is split exact, these generators are additionally projective. With the above Grothendieck topology, $\mathcal{E}$ is an additive ∞-site in the sense of [Pst23, §2]. Following [Pst23], we call an additive presheaf **spherical**. By the universal properties of $K_b(\mathcal{E})$ and $D_b(\mathcal{E})$, there are identifications

$$\mathrm{Ind}(K_b(\mathcal{E})) = \mathrm{Fun}^{ex}(K_b(\mathcal{E})^{op}, \mathcal{S}p) \simeq \mathrm{Fun}^{\oplus}(\mathcal{E}^{op}, \mathcal{S}p) = \mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E}),$$
$$\mathrm{Ind}(D_b(\mathcal{E})) = \mathrm{Fun}^{ex}(D_b(\mathcal{E})^{op}, \mathcal{S}p) \simeq \mathrm{Fun}^{ex}(\mathcal{E}^{op}, \mathcal{S}p) = \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E}),$$

where we use [Pst23, Th. 2.8] to identify spherical sheaves of spectra on $\mathcal{E}$ with those spherical presheaves sending congressions to fibre sequences of spectra. Note that the Yoneda embedding $\mathcal{E}_\oplus \hookrightarrow \mathcal{E}_\oplus^{lex} = \mathrm{Fun}^{add}(\mathcal{E}^{op}, \mathcal{A}b)$ induces by [Pst23, Lem. 2.61] an equivalence

$$\mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E}) \xrightarrow{\simeq} D(\mathcal{E}_\oplus^{lex})$$

between the the ∞-category of spherical presheaves of spectra on $\mathcal{E}$ and unbounded derived ∞-category of $\mathcal{E}_\oplus^{lex}$. Given a spherical sheaf of spectra X on $\mathcal{E}$, write $\pi_n^\dagger(X) : \mathcal{E}^{op} \to \mathcal{A}b$ for the sheaf associated to the assignment $c \mapsto \pi_n X(c)$, and call X connective if $\pi_n^\dagger(X) = 0$ for each $n < 0$, and coconnective if the sheaf of spaces $\Omega^\infty X$ given by postcomposition with $\Omega^\infty : \mathcal{S}p \to \mathcal{S}$ is discrete. By [Pst23, Prop. 2.16], the pair $(\mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})_{\leq 0}, \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})_{\geq 0})$ defines a t-structure on $\mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})$, for which we denote by $\tau_{\geq 0}$, $\tau_{\leq 0}$ the respective truncation functors. We refer to this as the **Postnikov** t-structure.

6.3 Genuine symmetric structures

For a hermitian category $(\mathcal{C}, \Omega)$, recall that the first excisive approximation $P_1 \Omega = \mathrm{colim}_n\, \Omega^n \Omega \Sigma^n$ is computed as the cofibre

$$[\Delta^* B_\Omega]_{hC_2} \to \Omega \to P_1 \Omega,$$

since by [CDH+I, Prop. 1.1.13] this cofibre is 1-excisive, P_1 is left exact, and by [CDH+I, Lem. 1.3.1] the functor $[\Delta^* B_\Omega]_{hC_2}$ has vanishing 1-excisive approximation. We may view $P_1 \Omega$ as an object of $\mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^{lex}(\mathcal{C}^{op}, \mathcal{S}) \simeq \mathrm{Fun}^{ex}(\mathcal{C}^{op}, \mathcal{S}p)$, and if $\mathcal{C}$ is equipped with an orientation $(\mathrm{Ind}_{\geq 0}(\mathcal{C}), \mathrm{Ind}_{\leq 0}(\mathcal{C}))$, we say that Ω is m*-connective* if $P_1 \Omega \in \mathrm{Ind}(\mathcal{C})$ is m-connective. By [CHN25, Lem. 3.3.1], the inclusion $\mathrm{Fun}^q_{\geq m}(\mathcal{C}) \subset \mathrm{Fun}^q(\mathcal{C})$ of m-connective quadratic functors with respect to this orientation admits a right adjoint $\Omega \mapsto \Omega \underset{P_1\Omega}{\times} \tau_{\geq m} P_1 \Omega$, where the connective cover is taken in $\mathrm{Ind}(\mathcal{C})$. Given an exact category with duality $(\mathcal{E}, \mathbb{D}, \eta)$, write $Q^s : \mathcal{E}^{op} \to \mathcal{A}b$ for the symmetric forms functor, and $\Omega^s : D_b(\mathcal{E})^{op} \to \mathcal{S}p$ for the (homotopy) symmetric Poincaré structure on $D_b(\mathcal{E})$, with linear part

$$\Lambda^s := \hom_{D_b(\mathcal{E})}(-, \mathbb{D}(-))^{tC_2}.$$

Write $(\mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})_{\geq 0}, \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})_{\leq 0})$ for the t-structure of the previous section, and for $m \in \mathbb{Z}$ write $\Omega^{\geq m} := \Omega^s \underset{\Lambda^s}{\times} \tau_{\geq m}\Lambda^s$ for the associated truncated hermitian structure.

Lemma 6.3.1. *For $(\mathcal{E}, \mathbb{D}, \eta)$ a weakly idempotent complete exact category with duality, the derived quadratic functor of Q^s on $D_b(\mathcal{E})$ coincides with the genuine symmetric Poincaré structure $\Omega^{\geq 0}$, with respect to the Postnikov t-structure.*

Proof. For $\mathcal{E} = \mathcal{E}_{\oplus}$ split exact, this follows from a connectivity argument as in [CDH⁺I, Rem. 4.2.16]. In this case, the t-structure on $\mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E}) = \mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$ has connective part the subcategory of spherical presheaves of spectra factoring through $\mathcal{S}p_{\geq 0} \subset \mathcal{S}p$, and since postcomposition with the connective cover $\tau_{\geq 0} : \mathcal{S}p \to \mathcal{S}p_{\geq 0}$ sends spherical presheaves of spectra to spherical presheaves of connective spectra, the Postnikov truncation is computed pointwise, i.e. $(\tau_{\geq 0}\mathcal{F})(x) = \tau_{\geq 0}(\mathcal{F}(x))$ for each $x \in \mathcal{E}$ and $\mathcal{F} \in \mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$. Accordingly, the bicartesian square of spectra

$$\begin{array}{ccc} \Omega^{\geq 0}(x) & \longrightarrow & \tau_{\geq 0}\left[\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{tC_2}\right] \\ \downarrow & & \downarrow \\ \hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{hC_2} & \longrightarrow & \hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{tC_2} \end{array}$$

induces fibre sequences

$$\Omega^{\geq 0}(x) \to \hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{hC_2} \to \tau_{\leq -1}\left[\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{tC_2}\right],$$
$$\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))_{hC_2} \to \Omega^{\geq 0}(x) \to \tau_{\geq 0}\left[\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{tC_2}\right]$$

for each $x \in \mathcal{E} \subset K_b(\mathcal{E})$. Since the full subcategory $\mathcal{E} \subset K_b(\mathcal{E})$ of discrete complexes is stable under the duality, $\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{hC_2}$ is coconnective and $\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))_{hC_2}$ connective, and the associated long exact sequence on homotopy groups implies that $\Omega^{\geq 0}(x)$ is concentrated in degree 0, with

$$\begin{aligned} \pi_0\Omega^{\geq 0}(x) &\cong \pi_0\Omega^s(x) \\ &= \pi_0\left[\hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))^{hC_2}\right] \\ &\cong \left[\pi_0 \hom_{K_b(\mathcal{E})}(x, \mathbb{D}(x))\right]^{C_2} \\ &\cong \mathrm{Hom}_{\mathcal{E}}(x, \mathbb{D}(x))^{C_2} = Q^s(x), \end{aligned}$$

and we conclude by [BGMN22, Th. 2.19].

For $\mathcal{E}$ a general exact category, write $\Omega^s_{\oplus} : K_b(\mathcal{E})^{op} \to \mathcal{S}p$ and $\Omega^s : D_b(\mathcal{E})^{op} \to \mathcal{S}p$ for the respective symmetric Poincaré structures, with truncations $\Omega^{\geq 0}_{\oplus}$ and $\Omega^{\geq 0}$. The statement for general $\mathcal{E}$ follows from the claim that the left Kan extension of $\tau_{\geq 0}\Omega_{\oplus}$ along the Verdier projection is equivalent to $\Omega^{\geq 0}$. By [HTT, Rem. 6.2.2.12], the sheafification functor $(-)^{\dagger} : \mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathcal{E}) \to \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$ is computed as a filtered colimit

$$\mathcal{F}^{\dagger}(x) = \operatorname*{colim}_{y \twoheadrightarrow x} \mathcal{F}(y),$$

and coincides with left Kan extension along π^{op} under the identifications $\mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathcal{E}) \simeq \mathrm{Ind}(K_b(\mathcal{E}))$ and $\mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E}) \simeq \mathrm{Ind}(D_b(\mathcal{E}))$. By [Pst23, Prop. 2.16], the inclusion of the coconnective aisle $\mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})_{\leq 0} \subset \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$ commutes with filtered colimits, so that the connective cover $\tau_{\geq 0} : \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E}) \to \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})_{\geq 0}$ commutes with filtered colimits. We thus have for each excisive functor $\Lambda : K_b(\mathcal{E})^{op} \to \mathcal{S}p$ that $\pi_! \tau_{\geq 0}\Lambda^s \simeq \tau_{\geq 0}\pi_!\Lambda$, and in particular

$$\pi_!\tau_{\geq 0}P_1\Omega_{\oplus} \simeq \tau_{\geq 0}\pi_!P_1\Omega^s_{\oplus} \simeq \tau_{\geq 0}P_1\Omega^s,$$

where the last equivalence follows from Lemma 6.1.2 and the fact that P_1 is a left adjoint. In particular, the fibre sequence

$$\pi_!\left(\hom_{K_b(\mathcal{E})}(-,\mathbb{D}(-))_{hC_2}\right)(x) \to \pi_!\left(\hom_{K_b(\mathcal{E})}(-,\mathbb{D}(-))^{hC_2}\right)(x) \to \pi_!\left(\hom_{K_b(\mathcal{E})}(-,\mathbb{D}(-))^{tC_2}\right)(x)$$

coincides for each $x \in D_b(\mathcal{E})$ with

$$\hom_{D_b(\mathcal{E})}(x,\mathbb{D}(x))_{hC_2} \to \hom_{D_b(\mathcal{E})}(x,\mathbb{D}(x))^{hC_2} \to \hom_{D_b(\mathcal{E})}(x,\mathbb{D}(x))^{tC_2} = P_1\Omega^s(x).$$

Now the cartesian square defining $\Omega^{\geq 0}_\oplus$ is stable under the filtered colimit computing left Kan extension along the Verdier projection $K_b(\mathcal{E}) \to D_b(\mathcal{E})$, and so

$$\pi_!\Omega^{\geq 0}_\oplus = \pi_!\left[\Omega^s_\oplus \underset{\Lambda^s_\oplus}{\times} \tau_{\geq 0}\Lambda^s_\oplus\right] \simeq [\pi_!\Omega^s_\oplus] \underset{[\pi_! P_1 \Omega^s_\oplus]}{\times} [\pi_!\tau_{\geq 0}P_1\Omega^s_\oplus] \simeq \Omega^{\geq 0},$$

and we are done. □

6.3.1 Genuine symmetric hermitian K-theory of qs divisorial schemes

Suppose X is a scheme admitting an ample family of line bundles in the sense of [TT90, Def. 2.1], i.e. for which there exists a finite set $\{\mathcal{L}_1,\dots,\mathcal{L}_n\}$ of line bundles on X and sections $s_i \in \mathcal{L}_i$ such that the non-vanishing loci $X_{s_i} := \{x \in X \mid (s_i)_x \neq 0 \in k(x) \otimes \mathcal{L}_{i,x}\}$ form an open affine cover of X (note that such a scheme is necessarily quasi-compact). We call such a scheme **divisorial**. Write QCoh_X for the Grothendieck abelian category of quasi-coherent $\mathcal{O}_X$-modules. Then by [TT90, Lem. 2.1.3], quasi-coherent sheaves have the resolution property: each $\mathcal{F} \in \mathrm{QCoh}_X$ admits an epimorphism from a vector bundle, which moreover can be chosen to be a direct sum of powers of the $\mathcal{L}_i$ (with repetition). Since each vector bundle in compact in QCoh_X, this implies that the subcategory $\mathrm{Vect}_X \subset \mathrm{QCoh}_X$ is a system of compact generators, and by [Pst23, Th. 2.58] the restricted Yoneda embedding $\mathrm{QCoh}_X \to \mathcal{P}^{\mathrm{Set}}_\Sigma(\mathrm{Vect}_X)$ induces an equivalence

$$\mathrm{QCoh}_X \simeq \mathcal{S}h^{\mathrm{Set}}_\Sigma(\mathrm{Vect}_X).$$

Write $D^{qc}(X)$ for the stable ∞-category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves in each degree. A quasi-coherent complex $\mathcal{F} \in D^{qc}(X)$ is compact if and only if it is perfect, i.e. locally equivalent to a bounded complex of vector bundles (see for instance [CHN25, Lem. A.5.3]), and we write $\mathrm{Perf}(X) := D^{qc}(X)^\omega$ for the full subcategory of such. There is an exact inclusion $\mathrm{Vect}_X \subset \mathrm{Perf}_X$, inducing, by the universal property of the bounded derived ∞-category, an exact functor

$$D_b(\mathrm{Vect}_X) \to \mathrm{Perf}_X,$$

which for X divisorial is an equivalence of stable ∞-categories by [Sch10b, Prop. 8]. Moreover, it is also shown in *loc. cit.* that the compact objects of $D(\mathrm{QCoh}_X)$ are precisely the bounded complexes of finite locally free modules, i.e. $D(\mathrm{QCoh}_X)^\omega \simeq D_b(\mathrm{Vect}_X)$. For quasi-separated divisorial X, [TT90, Prop. 2.3.1(d), Th. 2.4.3] then furnishes an equivalence of categories $D_b(\mathrm{Vect}_X) \simeq \mathrm{Perf}_X$, and taking ind-completions we have an equivalence

$$\mathcal{S}h^{\mathcal{S}p}_\Sigma(\mathrm{Vect}_X) \simeq \mathrm{Fun}^{\mathrm{ex}}(D_b(\mathrm{Vect}_X)^{\mathrm{op}},\mathcal{S}p) \simeq \mathrm{Ind}(D_b(\mathrm{Vect}_X)) \simeq \mathrm{Ind}(\mathrm{Perf}_X) \simeq D^{qc}(X).$$

For X quasi-separated and divisorial, we claim that this equivalence is t-exact for the Postnikov t-structure on the left-hand side, and the classical t-structure (inherited from $D(X)$) on the right-. For such X we have by the discussion above and [Pst23, Th. 2.64] equivalences

$$\widehat{\mathcal{S}h}^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) \simeq \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) \simeq D(\mathrm{QCoh}_X) \simeq D^{qc}(X), \tag{6.1}$$

and moreover by *loc. cit.*, the functors $\mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) \to \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X)$ resp. $D(\mathcal{P}^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X)) \to D(\mathcal{S}h^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X))$ are localisations at the classes of maps sent to isomorphisms by homotopy resp. homology sheaves $L\pi_k(-)$ resp. $LH_k(-)$, for $L : \mathcal{P}^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X) \to \mathcal{S}h^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X)$ the sheafification. Accordingly, the diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) & \xrightarrow{\simeq} & D(\mathcal{P}^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X)) \\ \downarrow & & \downarrow \\ \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) & \xrightarrow{\simeq} & D(\mathcal{S}h^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X)) \end{array}$$

commutes by the universal property of localisation. Since under the equivalence $\mathcal{P}^{\mathcal{S}p}_{\Sigma}(\mathrm{Vect}_X) \simeq D(\mathcal{P}^{\mathcal{S}et}_{\Sigma}(\mathrm{Vect}_X))$, the homotopy presheaves of a spherical presheaf of spectra correspond to the homology presheaves of the associated object of the derived category, a spherical sheaf of spectra is connective if and only if the corresponding object in the derived category is, so that the equivalence (6.1) is t-exact.

For $\mathcal{L} \in \mathcal{P}ic(X)$ a line bundle on X, inducing a duality $\vee_{\mathcal{L}} : \mathrm{Vect}_X^{op} \to \mathrm{Vect}_X$, $\mathcal{E} \mapsto [\mathcal{E}, \mathcal{L}]$, write $GW^s(X, \mathcal{L}) := GW(\mathrm{Vect}_X, Q^s_{\mathcal{L}})$ for the classical hermitian K-theory spectrum of on-the-nose symmetric forms on Vect_X with respect to the duality $\vee_{\mathcal{L}}$. For $\mathbb{D}_{\mathcal{L}} : \mathrm{Perf}_X^{op} \to \mathrm{Perf}_X$ the duality induced by the extended duality $\mathrm{Ch}_b(\mathrm{Vect}_X)^{op} \to \mathrm{Ch}_b(\mathrm{Vect}_X)$, $M \mapsto [M, \mathcal{L}[0]]$, and the identification $\mathrm{Perf}_X \simeq D_b(\mathrm{Vect}_X)$, write $GW^{gs}(\mathrm{Perf}_X, \mathcal{L})$ for the genuine symmetric hermitian K-theory spectrum of the Poincaré category $(\mathrm{Perf}_X, Ϙ^{gs}_{\mathbb{D}_{\mathcal{L}}})$. The above discussion yields the following.

Corollary 6.3.2. *For* X *quasi-separated and divisorial, the inclusion* $\mathrm{Vect}_X \subset \mathrm{Perf}_X$ *induces an equivalence of Grothendieck-Witt spectra*

$$GW(X, Q^s_{\mathcal{L}}) \to GW^{gs}(\mathrm{Perf}_X, \mathcal{L}).$$

Remark 6.3.3. The above shows that agreement of on-the-nose and genuine symmetric GW of schemes is a purely formal feature, rather than being an intrinsically geometric fact. The same statement for Grothendieck-Witt spaces was proven by Calmès-Harpaz-Nardin [CHN25, Prop. 4.6.1], using Zariski descent results of Schlichting and *ibid*.

Appendix A: Deriving exact categories

A.1 Exact ∞-categories

Exact categories were introduced by Quillen [Qui73] as a generalisation of abelian categories, in which enough of the usual diagram lemmas hold to permit a useful theory of homological algebra. Notable examples are additive (split-exact) categories, categories of algebraic vector bundles over schemes, abelian categories, and categories of endomorphisms. The 'minimal' axioms of Keller [Kel90] generalise readily to give an ∞-categorical formulation [Bar15], which we present below. Exact ∞-categories interpolate between additive and stable ∞-categories, these being the minimal and maximal examples of such.

Recall that an ∞-category $\mathcal{A}$ is semiadditive if it admits finite products and coproducts, has a zero object, and if the canonical map

$$\begin{pmatrix} \mathrm{id}_x & 0 \\ 0 & \mathrm{id}_y \end{pmatrix} : x \amalg y \to x \times y$$

is an equivalence for each $x, y \in \mathcal{A}$; in this case we denote by $x \oplus y := x \times y \simeq x \amalg y$ the biproduct, or direct sum. Mapping spaces in such categories canonically enhance to $\mathbb{E}_\infty$-spaces, and each object of $\mathcal{C}$ can be promoted canonically to an $\mathbb{E}_\infty$-monoid via the fold map $\nabla : x \oplus x \to x$. A semiadditive category $\mathcal{A}$ is said to be additive if for object x, the shear map $\begin{pmatrix} \mathrm{id}_x & \mathrm{id}_x \\ 0 & \mathrm{id}_x \end{pmatrix}$ is an equivalence, or equivalently, the mapping space bifunctor admits a lift to grouplike $\mathbb{E}_\infty$-spaces. In this case, the forgetful functors

$$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{A}) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{A}) \to \mathcal{A}$$

are equivalences, where we consider $\mathbb{E}_\infty$-monoids with respect to the cocartesian symmetric monoidal structure on $\mathcal{A}$, which by semiadditivity coincides with the cartesian structure. Alternatively, $\mathcal{A}$ is additive if and only if it admits finite products and coproducts, and $\mathrm{Ho}(\mathcal{A})$ is an ordinary additive category.

Example A.1.1. An ordinary additive category is an additive ∞-category. Any stable ∞-category is additive.

Definition A.1.2 (Quillen, Keller, Barwick)**.** An exact ∞-category is a tuple $(\mathcal{E}, \mathcal{E}_{\mathrm{in}}, \mathcal{E}_{\mathrm{eg}})$, for $\mathcal{E}$ an additive ∞-category, and $\mathcal{E}_{\mathrm{in}}, \mathcal{E}_{\mathrm{eg}} \subset \mathcal{E}$ are wide subcategories satisfying the following:

(i) For each x in $\mathcal{E}$, $0 \to x$ is in $\mathcal{E}_{\mathrm{in}}$ and $x \to 0$ in $\mathcal{E}_{\mathrm{eg}}$.

(ii) Pushouts along maps in $\mathcal{E}_{\mathrm{in}}$ exist and are in $\mathcal{E}_{\mathrm{in}}$, and dually pullbacks along maps in $\mathcal{E}_{\mathrm{eg}}$ exist and are in $\mathcal{E}_{\mathrm{eg}}$.

(iii) A square

$$\begin{array}{ccc} x & \xrightarrow{i} & y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ x' & \xrightarrow{i'} & y' \end{array}$$

in $\mathcal{E}$ is cocartesian with $i \in \mathcal{E}_{in}$ and $p \in \mathcal{E}_{eg}$ if and only if it is cartesian, with $i' \in \mathcal{E}_{in}$ and $p' \in \mathcal{E}_{eg}$.

Arrows in $\mathcal{E}_{in}$ are called **ingressive**, and denoted $\rightarrowtail$, and arrows in $\mathcal{E}_{eg}$ **egressive**, and denoted $\twoheadrightarrow$. A square as in (iii) is said to be **exact**, and an exact square

$$\begin{array}{ccc} x & \rightarrowtail & y \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ 0 & \rightarrowtail & z \end{array}$$

is said to be a congression, or an exact sequence. An additive functor $(\mathcal{E}, \mathcal{E}_{in}, \mathcal{E}_{eg}) \to (\mathcal{E}', \mathcal{E}'_{in}, \mathcal{E}'_{eg})$ is said to be exact if it preserves ingressions and pushouts along these, and egressions and pullbacks along these.

Remark A.1.3. Terminology for exact categories abounds: ingressions, egressions, and congressions (Barwick) are variously known as admissible monomorphisms, admissible epimorphisms, and admissible exact sequences (Quillen); inflations, deflations, conflations (Keller, Schlichting); cofibrations, fibrations, and exact sequences (Klemenc); and exact inclusions, exact projections, and exact sequences (Saunier). We refrain from adding to this list.

We will usually abuse notation and simply denote an exact ∞-category by its underlying ∞-category. Write $\mathrm{Fun}^{ex}(\mathcal{E}, \mathcal{E}') \subset \mathrm{Fun}(\mathcal{E}, \mathcal{E}')$ for the full subcategory of exact functors $\mathcal{E} \to \mathcal{E}'$. Then the collection of exact ∞-categories and exact functors assembles into an ∞-category $\mathcal{E}\mathrm{xact}_\infty$, obtained for instance as the homotopy coherent nerve of the simplicial category $\mathcal{E}\mathrm{xact}_\Delta$ with objects exact quasi-categories and $\mathrm{Map}_{\mathcal{E}\mathrm{xact}_\Delta}(\mathcal{E}, \mathcal{E}') := \mathrm{Fun}^{ex}(\mathcal{E}, \mathcal{E}')^{\simeq}$, the maximal sub-Kan complex of the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{E}')$ spanned by the exact functors.

Remark A.1.4. Suppose given a full additive subcategory $\mathcal{A} \subset \mathcal{E}$ of an exact ∞-category with essential image closed under extensions (i.e. for each exact sequence $x \rightarrowtail y \twoheadrightarrow z$, if $x, z \in \mathcal{A}$ then also $y \in \mathcal{A}$) and define $\mathcal{A}_{in}$ resp. A_{eg} to be the class of maps in $\mathcal{A}$ in $\mathcal{E}_{in}$ resp. $\mathcal{E}_{eg}$ with cofibre resp. fibre in $\mathcal{A}$. This defines an exact structure on $\mathcal{A}$ such that the inclusion functor is exact: a cospan $x' \overset{p}{\twoheadrightarrow} x \overset{f}{\leftarrow} y$ in which the left leg is in $\mathcal{A}_{eg}$ may be completed to a cartesian square

$$\begin{array}{ccc} y' & \xrightarrow{p'} & y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ x' & \overset{p}{\twoheadrightarrow} & x \end{array}$$

in $\mathcal{E}$, and one observes that p' is an egression in $\mathcal{E}$ with fibre in $\mathcal{A}$, so that by closure under extensions $y' \in \mathcal{A}$, and $p' \in \mathcal{A}_{eg}$. The dual argument holds for $\mathcal{A}_{in}$, and (iii) follows from the same axiom for $\mathcal{E}$. We will refer to this exact structure on $\mathcal{A}$ as the induced exact structure.

Example A.1.5. Any additive ∞-category $\mathcal{A}$ carries the split-exact structure, in which ingressions and egressions are the canonical inclusions into and projections from direct sums; for $\mathcal{C}$ a stable ∞-category, $\mathcal{C}_{in} = \mathcal{C} = \mathcal{C}_{eg}$ defines an exact structure on $\mathcal{C}$. An abelian category $\mathcal{A}$ has a canonical exact structure in which $\mathcal{A}_{in}$ is the class of monomorphisms, and $\mathcal{A}_{eg}$ the class of epimorphisms.

A.1.1 Stable envelopes

By a classical result of Gabriel and Quillen [TT90, Th. A.7.1], every ordinary exact category $\mathcal{E}$ admits an exact embedding into an abelian category, which moreover reflects exact sequences and has essential image closed under extensions; a canonical choice for such an embedding is the Grothendieck abelian category $\mathcal{P}_{\mathrm{lex}}^{\mathcal{A}b}(\mathcal{E})$ of left-exact presheaves of abelian groups on $\mathcal{E}$, equipped with the Yoneda embedding. This embedding has a higher categorical analogue due to work of Klemenc [Kle22], Saunier-Winges [SW25], and Nielsen-Winges [NW25], which we sketch below.

Recall first that to an ordinary exact category $\mathcal{E}$ we may associate its chain homotopy category $\mathrm{K}_b(\mathcal{E})$, modelled by the simplicial nerve of the category $\mathrm{Ch}_b(\mathcal{E})$ of bounded chain complexes over $\mathcal{E}$. The category $\mathrm{K}_b(\mathcal{E})$ is a stable ∞-category (Prop. A.3.13), and taking the Verdier quotient by the subcategory $\mathrm{Ac}_b(\mathcal{E})$ of acyclic complexes, we obtain the bounded derived ∞-category $\mathrm{D}_b(\mathcal{E})$, recovering the classical bounded derived category as its homotopy category. By [BCKW24, Cor. 7.4.12] restriction along the inclusion $\mathcal{E} \subset \mathrm{D}_b(\mathcal{E})$ induces an equivalence, for $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$,

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}_b(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}). \tag{A.1}$$

Note that $\mathrm{D}_b(\mathcal{E})^{\mathrm{op}} \simeq \mathrm{D}_b(\mathcal{E}^{\mathrm{op}})$ by comparing universal properties, so that the ind-completion $\mathrm{Ind}(\mathrm{D}_b(\mathcal{E}))$ identifies with $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{D}_b(\mathcal{E})^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}\mathrm{p})$.

The exact structure on $\mathcal{E}$ gives rise to a Grothendieck pretopology on $\mathcal{E}$, in which covers are singleton families of egressions $\{y \twoheadrightarrow x\}$. A presheaf $\mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is a sheaf for this topology precisely when it is left exact (see for instance the proof of [NW25, Th. 2.7]), and accordingly we have an identification $\mathrm{Ind}(\mathrm{D}_b(\mathcal{E})) = \mathcal{S}\mathrm{h}_\Sigma^{\mathcal{S}\mathrm{p}}(\mathcal{E})$ and an exact Yoneda embedding

$$\mathcal{E} \hookrightarrow \mathcal{S}\mathrm{h}_\Sigma^{\mathcal{S}\mathrm{p}}(\mathcal{E}).$$

The latter category is presentable and admits a Postnikov t-structure in which coconnectivity is detected levelwise, and connectivity is determined by homotopy sheaves. By [Pst23, Prop. 2.19], the connective aisle $\mathcal{S}\mathrm{h}_\Sigma^{\mathcal{S}\mathrm{p}}(\mathcal{E})_{\geq 0}$ identifies with the category $\mathcal{S}\mathrm{h}_\Sigma(\mathcal{E})$ of sheaves of spaces on $\mathcal{E}$.

In fact, all of the above holds for exact ∞-categories. The following is due to Nielsen-Winges.

Construction A.1.6. Let $\mathcal{E} \in \mathcal{E}\mathrm{xact}_\infty$, and write $\mathcal{P}_\Sigma(\mathcal{E})$ for the nonabelian derived category of product-preserving presheaves of spaces on $\mathcal{E}$. By [GGN15, Cor. 4.9], the underlying space functor induces an equivalence

$$\mathrm{Fun}^{\oplus}(\mathcal{E}^{\mathrm{op}}, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})) \to \mathrm{Fun}^{\times}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}_\Sigma(\mathcal{E}),$$

and the category $\mathcal{P}_\Sigma(\mathcal{E})$ is Grothendieck prestable. The full subcategory $\mathcal{P}_{\mathrm{lex}}(\mathcal{E}) \subset \mathcal{P}_\Sigma(\mathcal{E})$ of left-exact presheaves of connective spectra is a reflective subcategory, and the localisation $\Lambda : \mathcal{P}_\Sigma(\mathcal{E}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{E})$ preserves finite limits; we call $\mathcal{P}_{\mathrm{lex}}(\mathcal{E})$ the presentable envelope of $\mathcal{E}$, and $\mathcal{P}_{\mathrm{lex}}^{\mathrm{st}}(\mathcal{E}) := \mathcal{S}\mathrm{p}(\mathcal{P}_{\mathrm{lex}}(\mathcal{E}))$ the presentable stable envelope.

By [NW25, Th. 2.10], the (enriched) Yoneda embedding then identifies $\mathcal{E}$ as an extension-closed subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{E})$, and restriction along this induces, for any pointed cocomplete $\mathcal{C}$, an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}),$$

where the latter category is the subcategory of exact functors of Waldhausen ∞-categories (for the maximal Waldhausen structure $\mathcal{C} = \mathcal{C}_{\mathrm{in}}$), i.e. those functors preserving zero objects, ingressions and pushouts along these.

For $\mathcal{C}$ presentable and stable, we accordingly have an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}).$$

Using the above, we may recover Klemenc's stable hull [Kle22] : write $\mathrm{St}_{\geq 0}(\mathcal{E}) \subset \mathcal{P}_{\mathrm{lex}}(\mathcal{E})$ for the smallest full prestable subcategory containing the essential image of the Yoneda embedding, and define $\mathrm{St}(\mathcal{E}) := \mathcal{SW}(\mathrm{St}_{\geq 0}(\mathcal{E}))$, for $\mathcal{SW}(\mathcal{C}) := \mathrm{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \dots)$ the Spanier-Whitehead stabilisation, with the colimit taken in $\mathcal{C}\mathrm{at}_\infty$. Restriction along $\mathcal{E} \to \mathrm{St}(\mathcal{E})$ induces an equivalence, for each $\mathcal{C} \in \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$,

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}),$$

and in particular St is a left adjoint to the inclusion $\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty \subset \mathcal{E}\mathrm{xact}_\infty$, coinciding with D_b on the full subcategory of ordinary exact categories. There is a commutative square of full inclusions

$$\begin{array}{ccc} \mathrm{St}_{\geq 0}(\mathcal{E}) & \hookrightarrow & \mathrm{St}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{P}_{\mathrm{lex}}(\mathcal{E}) \simeq \mathrm{Ind}(\mathrm{St}_{\geq 0}(\mathcal{E})) & \hookrightarrow & \mathrm{Ind}(\mathrm{St}(\mathcal{E})) \simeq \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{E}). \end{array}$$

A.1.2 Symmetric monoidal structure

Write $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ for the ∞-category of presentable ∞-categories and colimit-preserving functors. By [HA, Prop. 4.8.1.15], $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a symmetric monoidal structure with unit the ∞-category of spaces, with $\mathcal{C} \otimes \mathcal{D}$ the initial presentable ∞-category receiving a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ preserving small colimits in each variable separately, so

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{biL}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Write $\mathcal{P}\mathrm{r}^{\mathrm{add}} \subset \mathcal{P}\mathrm{r}^{\mathrm{L}}$ for the full subcategory of additive presentable ∞-categories. Then by [Lur18, Cor. C.4.1.3, C.4.1.4], there is an identification $\mathcal{P}\mathrm{r}^{\mathrm{add}} \simeq \mathrm{Mod}_{\mathcal{S}\mathrm{p}_{\geq 0}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, and the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{add}} \subset \mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a left adjoint $\mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{add}}$, inducing a unique symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{add}}$ such that L promotes to a symmetric monoidal functor: indeed, L is a smashing localisation given by tensoring with $\mathcal{S}\mathrm{p}_{\geq 0}$. The full subcategory $\mathcal{G}\mathrm{roth}_\infty \subset \mathcal{P}\mathrm{r}^{\mathrm{add}}$ spanned by the Grothendieck prestable ∞-categories is closed under this symmetric monoidal structure by [Lur18, Th. C.4.2.1].

Construction A.1.7 ([NW25, Cons. 3.4])**.** Given $\mathcal{C}, \mathcal{D} \in \mathcal{E}\mathrm{xact}_\infty$, consider the composite

$$y : \mathcal{C} \times \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \times \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}),$$

and define $\mathcal{C} \otimes \mathcal{D}$ to be the smallest full subcategory of the target containing the essential image of y and closed under extensions. We have by definition a square

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{j \times j} & \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \times \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \\ \downarrow t & & \downarrow \\ \mathcal{C} \otimes \mathcal{D} & \xrightarrow{i} & \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}). \end{array}$$

We view $\mathcal{C} \otimes \mathcal{D}$ as an exact ∞-category by equipping it with the induced structure of Remark A.1.3.

By [NW25, Prop. 3.5, 3.7], this tensor product underlies a symmetric monoidal structure on $\mathcal{E}\mathrm{xact}_\infty$, and $\mathcal{P}_{\mathrm{lex}}$ promotes to a symmetric monoidal functor: given exact ∞-categories $\mathcal{C}$, $\mathcal{D}$, one defines an exact structure on $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ with $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})_{\mathrm{in}}$ given by those natural transformations $F \Rightarrow G$ which are objectwise ingressions in $\mathcal{D}$. The cartesian closedness of $\mathcal{C}\mathrm{at}_\infty$ induces equivalences, for each $\mathcal{E} \in \mathcal{G}\mathrm{roth}_\infty$,

$$\begin{aligned}\mathrm{Fun}^{\mathrm{biex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}))\\ &\simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{D}), \mathcal{E}))\\ &\simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}\, D, \mathcal{E}),\end{aligned}$$

where the final two equivalences follow from the universal properties of $\mathcal{P}_{\mathrm{lex}}$ and the tensor product on $\mathcal{G}\mathrm{roth}_\infty$. The functor $i : \mathcal{C} \otimes \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$ extends uniquely to $T : \mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$, and the biexact functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D})$ induces a functor $I : \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{D})$, inverse to T.

Remark A.1.8. Write $\mathcal{A}\mathrm{dd}_\infty$ for the ∞-category of additive ∞-categories and additive functors. There is an adjunction

$$\mathcal{A}\mathrm{dd}_\infty \underset{U}{\overset{j}{\rightleftarrows}} \mathcal{E}\mathrm{xact}_\infty,$$

where U forgets the underlying exact structure (but remembers additivity), and j equips an additive category with the split exact structure, in which the exact inclusions resp. projections are the canonical inclusions into resp. projections from direct sums. The category of exact functors between $\mathcal{C}, \mathcal{D} \in \mathcal{E}\mathrm{xact}_\infty$ acquires an exact structure in which a map $\tau : F \Rightarrow G$ is an exact inclusion or projection if it is so pointwise in $\mathcal{D}$. The adjunction $j \dashv U$ extends to the appropriate functor categories, and this is an additive equivalence of additive categories, i.e.

$$U\,\mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, U(\mathcal{E})) \in \mathcal{A}\mathrm{dd}_\infty.$$

Note that an exact category has the split exact structure precisely when it is local with respect to the counit map $j(U(\mathcal{E})) \to \mathcal{E}$ for each exact $\mathcal{E}$,

$$\mathrm{Map}_{\mathcal{E}\mathrm{xact}_\infty}(j(\mathcal{A}), jU(\mathcal{E})) \simeq \mathrm{Map}_{\mathcal{E}\mathrm{xact}_\infty}(j(\mathcal{A}), \mathcal{E}). \tag{A.2}$$

While it does not follow that the map

$$\mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), jU(\mathcal{E})) \to \mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), \mathcal{E})$$

induced by the counit is an equivalence of exact categories (the exact structures differ), this equivalence does hold on underlying additive categories, i.e.

$$U\,\mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), jU(\mathcal{E})) = U\,\mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), \mathcal{E}). \tag{A.3}$$

It is also not true the counit

$$jU\,\mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), jU(\mathcal{E})) \to \mathrm{Fun}^{\mathrm{ex}}(j(\mathcal{A}), jU(\mathcal{E}))$$

is an equivalence of exact categories, since a natural transformation which splits pointwise doesn't need to globally, but $j(\mathcal{A})$ doesn't notice this since it is local with respect to all such counits.

Now $\mathcal{E}\mathrm{xact}_\infty$ has a tensor product equipping it with a closed symmetric monoidal structure. Given $\mathcal{A}, \mathcal{B} \in \mathcal{A}\mathrm{dd}_\infty$ and some exact ∞-category $\mathcal{E}$, the chain of equivalences (of **additive** categories)

$$\begin{aligned}
\mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}) \otimes \mathrm{j}(\mathcal{B}), \mathcal{E}) &\simeq \mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}), \mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathcal{E})) \\
&\overset{(A.3)}{\simeq} \mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}), \mathrm{jU}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathcal{E})) \\
&\overset{(A.2)}{\simeq} \mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}), \mathrm{jU}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathrm{jU}(\mathcal{E}))) \\
&\overset{(A.3)}{\simeq} \mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}), \mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathrm{jU}(\mathcal{E}))) \\
&\simeq \mathrm{U}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{A}) \otimes \mathrm{j}(\mathcal{B}), \mathrm{jU}(\mathcal{E}))),
\end{aligned}$$

exhibits $\mathrm{j}(\mathcal{A}) \otimes \mathrm{j}(\mathcal{B})$ as split, and one traces through the equivalences to see that this is indeed induced by the counit $\mathrm{jU}(\mathcal{E}) \to \mathcal{E}$: modulo messing with exact structures, we are just currying, applying the counit, and uncurrying, using commutativity of the square

$$\begin{array}{ccc}
\mathrm{jU}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathrm{jU}(\mathcal{E})) & \xrightarrow{\varepsilon_{\mathrm{Fun}}} & \mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathrm{jU}(\mathcal{E})) \\
\downarrow{\scriptstyle \mathrm{jU}(\varepsilon_{\mathcal{E}} \circ -)} & & \downarrow{\scriptstyle \varepsilon_{\mathcal{E}} \circ -} \\
\mathrm{jU}\,\mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathcal{E}) & \xrightarrow{\varepsilon_{\mathrm{Fun}}} & \mathrm{Fun}^{\mathrm{ex}}(\mathrm{j}(\mathcal{B}), \mathcal{E}).
\end{array}$$

Accordingly, the symmetric monoidal structure on $\mathcal{E}\mathrm{xact}_\infty$ restricts to a symmetric monoidal structure on $\mathcal{A}\mathrm{dd}_\infty$, and $\mathcal{P}_{\mathrm{lex}}$ and St restrict to symmetric monoidal functors $\mathcal{P}_\Sigma : \mathcal{A}\mathrm{dd}_\infty \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathrm{St}^{\mathrm{add}} : \mathcal{A}\mathrm{dd}_\infty \to \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$.

Remark A.1.9. The stable envelope furnishes a spectral enrichment of the mapping spaces of an exact ∞-category $\mathcal{E}$, via restriction of the mapping spectra of $\mathrm{St}(\mathcal{E})$; write $\hom_{\mathcal{E}}(-,-) : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}\mathrm{p}$ for the bifunctor thus obtained. Recall [Gla16, Prop. 2.3] that the space of natural transformations $\mathrm{F} \Rightarrow \mathrm{G} : \mathcal{C} \to \mathcal{D}$ is computed as the end of the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{F}^{\mathrm{op}} \times \mathrm{G}} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Map}_{\mathcal{D}}(-,-)} \mathcal{S}$, i.e. as the limit

$$\int_{\mathcal{C}} \mathrm{Map}(\mathrm{F}(-)), \mathrm{G}(-)) = \lim_{\mathrm{TwAr}(\mathcal{C})} \mathrm{Map}_{\mathcal{D}}(\mathrm{F}(-), \mathrm{G}(-)).$$

Given exact ∞-categories $\mathcal{C}, \mathcal{D}$, write $\beta : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ for the canonical projection. For $x \otimes y := \beta((x, y))$, we claim that the mapping spectrum $\hom_{\mathcal{C}\otimes\mathcal{D}}(-, x \otimes y)$ is computed as the left Kan extension along β^{op} of the smash product $\hom_{\mathcal{C}}(-, x) \otimes \hom_{\mathcal{D}}(-, y)$: indeed, for $\mathrm{G} : (\mathcal{C} \otimes \mathcal{D})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ an exact functor, we have natural equivalences

$$\begin{aligned}
\mathrm{Nat}_{[(\mathcal{C}\otimes\mathcal{D})^{\mathrm{op}},\mathcal{S}\mathrm{p}]}(\beta_!(\hom_{\mathcal{C}}(-,x) \otimes \hom_{\mathcal{D}}(-,y)), \mathrm{G}) &\simeq \mathrm{Nat}_{[(\mathcal{C}\times\mathcal{D})^{\mathrm{op}},\mathcal{S}\mathrm{p}]}(\hom_{\mathcal{C}}(-,x) \otimes \hom_{\mathcal{D}}(-,y), \beta^*\mathrm{G}) \\
&\simeq \lim_{\mathrm{TwAr}(\mathcal{C}\times\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{C}}(-,x) \otimes \hom_{\mathcal{D}}(-,y), \beta^*\mathrm{G}) \\
&\simeq \lim_{\mathrm{TwAr}(\mathcal{C}\times\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{C}}(-,x), \hom_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{D}}(-,y), \beta^*\mathrm{G})) \\
&\simeq \lim_{\mathrm{TwAr}(\mathcal{C})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{C}}(-,x), \lim_{\mathrm{TwAr}(\mathcal{D})^{\mathrm{op}}} \hom_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{D}}(-,y), \beta^*\mathrm{G})) \\
&\simeq \mathrm{Nat}_{[\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{p}]}(\hom_{\mathcal{C}}(-,x), \lim_{\mathrm{TwAr}(\mathcal{D})^{\mathrm{op}}} \hom_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{D}}(-,y), \beta^*\mathrm{G})) \\
&\simeq \mathrm{Nat}_{[\mathrm{St}(\mathcal{C})^{\mathrm{op}},\mathcal{S}]}(\mathrm{Map}_{\mathrm{St}(\mathcal{C})}(-,x), \lim_{\mathrm{TwAr}(\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{D}}(-,y), \mathrm{G}')) \\
&\simeq \lim_{\mathrm{TwAr}(\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\hom_{\mathcal{D}}(-,y), \mathrm{G}'(x \otimes -)) \\
&\simeq \mathrm{Nat}_{[\mathcal{D}^{\mathrm{op}},\mathcal{S}\mathrm{p}]}(\hom_{\mathcal{D}}(-,y), \mathrm{G}'(x \otimes -)) \\
&\simeq \mathrm{Nat}_{[\mathrm{St}(\mathcal{D})^{\mathrm{op}},\mathcal{S})]}(\mathrm{Map}_{\mathrm{St}(\mathcal{D})}(-,y), \Omega^\infty \mathrm{G}''(x \otimes -)) \\
&\simeq \Omega^\infty \mathrm{G}''(x \otimes y)) \simeq \Omega^\infty \mathrm{G}(x \otimes y),
\end{aligned}$$

where we use the equivalence $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{St}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{St}(\mathcal{E})^{\mathrm{op}}, \mathcal{S})$ for any exact ∞-category $\mathcal{E}$, and we write $G' : \mathrm{St}(\mathcal{C})^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ and $G'' : \mathrm{St}(\mathcal{C})^{\mathrm{op}} \times \mathrm{St}(\mathcal{D})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ for the unique biexact extensions of $\beta^* G$. By Yoneda, we then have natural equivalences $\beta_!(\hom_{\mathcal{C}}(-,x) \otimes \hom_{\mathcal{D}}(-,y)) \simeq \hom_{\mathcal{C}\otimes\mathcal{D}}(-, x \otimes y)$, and dually $\beta_!(\hom_{\mathcal{C}}(x,-) \otimes \hom_{\mathcal{D}}(y,-)) \simeq \hom_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y, -)$. This generalises the usual formula for $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$.

1. Given two ordinary exact categories $\mathcal{C}, \mathcal{D}$ (for which the mapping spectra will in general be coconnective deloopings of the discrete mapping spaces), the mapping spectra of the tensor product $\mathcal{C} \otimes \mathcal{D}$ are no longer guaranteed to be coconnective even on objects in the essential image of $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$, being computed as a filtered colimit of smash products of coconnective spectra[1]. The full subcategories $\mathcal{A}\mathrm{dd}_1 \subset \mathcal{A}\mathrm{dd}_\infty$ and $\mathcal{E}\mathrm{xact}_1 \subset \mathcal{E}\mathrm{xact}_\infty$ are then somewhat unsurprisingly not closed under tensor products.

2. For $\mathcal{A}$ an exact ∞-category, we claim that $\mathcal{A}$ is split if and only if the induced mapping spectra $\hom_{\mathcal{A}}(x,y)$ are connective for each $x, y \in \mathcal{A}$: for $\mathcal{A}$ split, this follows since the stable envelope $\mathrm{St}(\mathcal{A}) = \mathrm{St}^{\mathrm{add}}(\mathcal{A})$ carries a weight structure. Conversely, given a congression $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{A}$ inducing an exact sequence in $\mathrm{St}(\mathcal{A})$, we note from the shifted sequence $y \to z \xrightarrow{\partial} \Sigma x$ that $\pi_0 \partial \in \pi_0 \hom_{\mathrm{St}(\mathcal{A})}(z, \Sigma x) \cong \pi_{-1} \hom_{\mathrm{St}(\mathcal{A})}(z, x) = 0$, so that ∂ is nullhomotopic, and the exact sequence splits in $\mathrm{St}(\mathcal{A})$ and hence $\mathcal{A}$. The computation of mapping spectra above then gives another argument that $\mathcal{A} \otimes \mathcal{B}$ is split.

A.2 Diagram lemmas

In this section we collate some diagram lemmas for exact ∞-categories; these are usually proved with straightforward generalisations of the corresponding arguments for ordinary exact categories, for which [Büh10] is an excellent reference. We claim no originality.

Lemma A.2.1 ([Kle22, Lem. A.1], [Bar15, Lem. 4.7]). *Let $\mathcal{E}$ an exact ∞-category, and consider a diagram*

$$\begin{array}{ccc} x & \xrightarrow{p} & y \\ \downarrow & & \downarrow \\ x' & \xrightarrow{p'} & y. \end{array}$$

Then the following are equivalent

(i) the square is bicartesian;

(ii) the square is cartesian;

(iii) the induced map $\mathrm{fib}(p) \to \mathrm{fib}(p')$ *is an equivalence.*

Corollary A.2.2. *Suppose given a sequence $x \overset{p}{\twoheadrightarrow} y \overset{q}{\twoheadrightarrow} z$ of egressions in an exact ∞-category. Then the induced sequence of fibres*

$$\mathrm{fib}(p) \to \mathrm{fib}(pq) \to \mathrm{fib}(q)$$

is a congression.

[1] Since $\mathcal{C} \otimes \mathcal{D}$ is generated by the essential image of $\mathcal{C} \times \mathcal{D} \to \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{D})$ under extensions and $\mathcal{E} \to \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{E})$ is fully faithful for each exact ∞-category $\mathcal{E}$, given $z \in \mathcal{C} \otimes \mathcal{D}$, the mapping spectra $\hom_{\mathcal{C}\otimes\mathcal{D}}(-, z)$ are iterated extensions of mapping spectra at the 'elementary tensors'.

Proof. In the diagram

$$\begin{array}{ccccc}
\mathrm{fib}(p) & \rightarrowtail & \mathrm{fib}(qp) & \longrightarrow & \mathrm{fib}(q) \\
\Vert & & \downarrow & & \downarrow \\
\mathrm{fib}(p) & \rightarrowtail & x & \overset{p}{\twoheadrightarrow} & y \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle qp} \\
 & & z & = & z,
\end{array}$$

the upper right square is bicartesian by the (dual of the) lemma, so that $\mathrm{fib}(qp) \twoheadrightarrow \mathrm{fib}(q)$ is an egression. Since the upper sequence is a fibre sequence, it is a congression. □

Corollary A.2.3 ([Kle22, Prop. A.2])**.** *Given a map of congressions* $(f, g, h) : (x \rightarrowtail y \twoheadrightarrow z) \to (x' \rightarrowtail y' \twoheadrightarrow z')$ *in an exact* ∞*-category, there is a factorisation*

$$\begin{array}{ccccc}
x & \rightarrowtail & y & \twoheadrightarrow & z \\
\downarrow{\scriptstyle f} & \ulcorner & \downarrow & & \Vert \\
x' & \rightarrowtail & w & \twoheadrightarrow & z \\
\Vert & & \downarrow & \lrcorner & \downarrow{\scriptstyle h} \\
x' & \rightarrowtail & y' & \twoheadrightarrow & z',
\end{array}$$

with the upper left and lower right squares cocartesian and cartesian respectively, and the square

$$\begin{array}{ccc}
y & \twoheadrightarrow & z \\
\downarrow & & \downarrow \\
y' & \twoheadrightarrow & z'
\end{array}$$

is Reedy fibrant (the map $y \to y' \times_{z'} z$ *is an egression) if and only if* f *is an egression.*

Proof. Form the pushout $w := x' \cup_x y$, with $w \to z$ the map induced by the pair $(y \twoheadrightarrow z, x' \xrightarrow{0} z)$. Then $w \to z$ is a cofibre of $x \rightarrowtail w$, and accordingly an egression. The map $w \to y'$ is induced by the pair $(x' \rightarrowtail y', y \xrightarrow{g} y')$, so that the lower right square commutes, and is cartesian by the previous lemma. For the final claim, f is egressive if and only if the upper left square is exact, if and only if $y \to w$ is an egression. □

Lemma A.2.4. *Given a pointwise egression of congressions*

$$\begin{array}{ccccc}
x & \rightarrowtail & y & \twoheadrightarrow & z \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
x' & \rightarrowtail & y' & \twoheadrightarrow & z',
\end{array}$$

the induced sequence of fibres $\mathrm{fib}(f) \to \mathrm{fib}(g) \to \mathrm{fib}(h)$ *is a congression.*

Proof. Clearly the sequence is a fibre sequence, and by the corollary, the rightmost square is Reedy fibrant, so that we have a sequence of egressions $y \twoheadrightarrow y' \times_{z'} z \twoheadrightarrow y'$, and a congression of associated fibres

$$\mathrm{fib}(y \twoheadrightarrow y' \times_{z'} z) \rightarrowtail \mathrm{fib}(g) \to \mathrm{fib}(y' \times_{z'} z \twoheadrightarrow y').$$

The map $y' \times_{z'} z \to z$ induces an equivalence $\mathrm{fib}(y' \times_{z'} z \to y') \simeq \mathrm{fib}(h)$, and $\mathrm{fib}(y \twoheadrightarrow y' \times_{z'} z)$ identifies with the total fibre $\mathrm{fib}(\mathrm{fib}(g) \to \mathrm{fib}(h))$, inducing a congression

$$\mathrm{fib}(f) \rightarrowtail \mathrm{fib}(g) \twoheadrightarrow \mathrm{fib}(h).$$

□

A.3 Complicial exact categories

In this section we review some results on complicial exact categories [Sch11, Def. 3.2.2]. These are similar in spirit to the complicial biWaldhausen categories of Thomason-Trobaugh [TT90], the notable example is being that of (sub)categories of chain complexes over an exact category. We start with following observation.
Any additive category admits a canonical action of the closed symmetric monoidal category of finitely generated free $\mathbb{Z}$-modules $\mathrm{Proj}_{\mathbb{Z}}$, encoding the direct sum. If $\mathcal{E}$ is exact, this promotes by virtue of the fact that congressions are stable under direct sum to a biexact functor $\otimes : \mathrm{Proj}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$ (equipping $\mathrm{Proj}_{\mathbb{Z}}$ with the split-exact structure). One can ask whether this action extends via the canonical inclusion $\mathrm{Proj}_{\mathbb{Z}} \hookrightarrow \mathrm{Ch}_b(\mathrm{Proj}_{\mathbb{Z}}) := \mathscr{C}_{\mathbb{Z}}$ to an action of the symmetric monoidal category of bounded complexes over $\mathrm{Proj}_{\mathbb{Z}}$.
The canonical example of such an extension is the action $\mathscr{C}_{\mathbb{Z}} \times \mathrm{Ch}_b(\mathcal{E}) \to \mathrm{Ch}_b(\mathcal{E})$, defined as follows: for bounded complexes $K \in \mathscr{C}_{\mathbb{Z}}$ and $X \in \mathrm{Ch}_b(\mathcal{E})$, $K \otimes X$ is the complex given in degree n by

$$(K \otimes X)_n = \bigoplus_i K_i \otimes X_{n-i},$$

with differential given on the summand $K_i \otimes X_{n-i}$ by $d_i^K \otimes \mathrm{id}_{X_{n-i}} + (-1)^i \otimes d_{n-i}^X$. Given maps $A : K \to L$ in $\mathscr{C}_{\mathbb{Z}}$ and $f : X \to Y$ in $\mathrm{Ch}_b(\mathcal{E})$, where A^* is degreewise the matrix $(a^n_{\alpha\beta})_{\alpha\beta} \in \mathrm{Mat}_{l_n \times k_n}(\mathbb{Z})$, the induced map $K \otimes X \to L \otimes Y$ is given in degree n on the summand $K_i \otimes X_{n-i} = X_{n-i}^{\oplus k_i}$ by the matrix $(a^i_{\alpha\beta} \cdot f_{n-i}) : X^{\oplus k_i} \to Y^{\oplus l_i}$. Since the tensor product of complexes preserves connectivity, this also restricts to an action $\mathscr{C}_{\mathbb{Z},\geq 0} \times \mathrm{Ch}_{b,\geq 0}(\mathcal{E}) \to \mathrm{Ch}_{b,\geq 0}(\mathcal{E})$.

Notation A.3.1. Write $\mathbb{1} := \mathbb{Z}[0]$ for the unit of the monoidal structure on $\mathscr{C}_{\mathbb{Z}}$, and C for the complex $0 \to \mathbb{Z} = \mathbb{Z} \to 0$ in degrees $[0,1]$, sitting in the congression $\mathbb{1} \rightarrowtail C \twoheadrightarrow T$. Dually, for $P := C[-1]$, we have a congression $\Omega \rightarrowtail P \twoheadrightarrow \mathbb{1}$. Write $[-,-]$ for the mapping complex in $\mathscr{C}_{\mathbb{Z}}$, given degreewise by

$$[X, Y]_n = \prod_i \mathrm{Hom}_{\mathrm{Proj}_{\mathbb{Z}}}(X_i, Y_{i+n}),$$

with differential $\partial(f) = d \circ f - (-1)^{|f|} f \circ d$. The mapping cone of $f : X \to Y$ is $C(f)_n = Y_n \oplus X_{n-1}$, with differential

$$d_n^{C(f)} = \begin{pmatrix} d_n & f_{n-1} \\ 0 & -d_{n-1} \end{pmatrix} : C(f)_n \to C(f)_{n-1};$$

this is isomorphic to the pushout of the span $CX \leftarrowtail X \xrightarrow{f} Y$, for CX the mapping cone of id_X. The closed symmetric monoidal structure $(\mathscr{C}_{\mathbb{Z}}, \otimes, [-,-], \mathbb{1})$ induces for each $n \in \mathbb{Z}$ a duality $K \mapsto [K, \mathbb{1}[n]]$, with double dual identification $\mathrm{can} : K \to [[K, \mathbb{1}[n]], \mathbb{1}[n]]$ the adjunct of the evaluation map $\mathrm{ev}_{\mathbb{1}[n]} : [K, \mathbb{1}[n]] \otimes K \to \mathbb{1}[n]$.
Given a duality $(\mathbb{D}, \eta)$ on $\mathcal{E}$, for a map of complexes $X \to Y$ in $\mathrm{Ch}_b(\mathcal{E})$ we make the following choices for the extended duality:

$$\mathbb{D}(X)_n = \mathbb{D}(X_{-n}), \quad \mathbb{D}(f)_n = \mathbb{D}(f_{-n}), \quad d_n^{\mathbb{D}(X)} = (-1)^{n-1}\, \mathbb{D}(d^X_{-n+1}), \quad (\eta_X)_n = (-1)^n \eta_{X_n}.$$

Now the category $\mathrm{Ch}_b(\mathcal{E})$ inherits a canonical exact structure from $\mathcal{E}$ in which congressions are defined degreewise, and moreover comes equipped with a notion of weak equivalence as follows. A complex X is said to be strictly acyclic if the differentials admit factorisations $d_n : X_n \overset{p_n}{\twoheadrightarrow} Z_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_{n-1}$, such that the sequences $Z_n \overset{i_n}{\rightarrowtail} X_n \overset{p_n}{\twoheadrightarrow} Z_{n-1}$ are congressions in $\mathcal{E}$ for each n; X is acyclic if chain homotopy equivalent to a strictly acyclic complex. If $\mathcal{E}$ is weakly idempotent complete (i.e. any retract admits a kernel), the classes of acyclic and strictly acyclic complexes coincide [Büh10, Prop. 10.14]. A map of complexes $X \to Y$ is said to be a **quasi-isomorphism** if its mapping cone is acyclic. The degreewise congressions and quasi-isomorphisms equip $\mathrm{Ch}_b(\mathcal{E})$ with the structure of an exact category with weak equivalences, and the tensor product $\otimes : (\mathscr{C}_{\mathbb{Z}}, \mathtt{ch.htp.}) \times (\mathrm{Ch}_b(\mathcal{E}), \mathtt{qis}) \to (\mathrm{Ch}_b(\mathcal{E}), \mathtt{qis})$ is an exact functor of such categories, where $\mathscr{C}_{\mathbb{Z}}$ is equipped with the (degreewise split-)exact structure inherited in this manner from $\mathrm{Proj}_{\mathbb{Z}}$, and the class of corresponding quasi-isomorphisms, which coincides with the chain homotopy equivalences. The symmetric monoidal structure on $\mathscr{C}_{\mathbb{Z}}$ is compatible with this structure in the sense that the tensor product preserves chain homotopy equivalences in each variable and is biexact; we call such a category a symmetric monoidal exact category with weak equivalences, and note that the inclusion $\mathrm{Proj}_{\mathbb{Z}} \hookrightarrow \mathscr{C}_{\mathbb{Z}}$ is symmetric monoidal.

Definition A.3.2. An (ordinary) exact category with weak equivalences is a pair $(\mathcal{E}, w)$, for $\mathcal{E}$ an ordinary exact category and $w \subset \mathcal{E}$ a wide subcategory of weak equivalences, satisfying 2-of-3, closed under isomorphisms and retracts in $\mathrm{Ar}(\mathcal{E})$, and stable under pushouts resp. pullbacks along ingressions resp. egressions. An exact functor $(\mathcal{D}, v) \to (\mathcal{E}, w)$ of exact categories with weak equivalences is an exact functor of ordinary exact categories that additionally sends v to w.

Definition A.3.3. A *complicial* structure on an exact category with weak equivalences $(\mathcal{E}, w)$ is the data of a biexact action of the symmetric monoidal exact category with weak equivalences $(\mathscr{C}_{\mathbb{Z}}, \mathtt{ch.htp}, \otimes, \mathbb{1})$, i.e. a functor

$$\otimes : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$$

which is exact in each variable as a functor of exact categories with weak equivalences, and is coherently unital and associative in the sense of [Gra76]. In particular, we note that given maps $f : K \to L$ in $\mathscr{C}_{\mathbb{Z}}$ and $p : x \to y$ in $\mathcal{E}$, the induced map $f \otimes p : K \otimes x \to L \otimes y$ lies in w as soon as f is a chain homotopy equivalence and $p \in w$. An exact functor of complicial exact categories with weak equivalences $f : (\mathcal{E}, w) \to (\mathcal{E}', w')$ is simply an exact functor of the underlying exact categories with weak equivalences; f is said to be complicial exact if it is compatible with the action of $\mathscr{C}_{\mathbb{Z}}$ in the sense that there is a natural isomorphism rendering the diagram

$$\begin{array}{ccc} \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} & \xrightarrow{\otimes} & \mathcal{E} \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle f} \\ \mathscr{C}_{\mathbb{Z}} \times \mathcal{E}' & \xrightarrow{\otimes'} & \mathcal{E}' \end{array}$$

commutative, satisfying the constraints of [Gra76]. We will generally abuse terminology by referring to an exact category with a complicial structure as a complicial exact category. For $K \in \mathscr{C}_{\mathbb{Z}}$ and $x \in \mathcal{E}$, we may write $Kx := K \otimes x$.

We may refine the above definition to the case where $(\mathcal{E}, w, \mathbb{D}, \eta)$ is an exact category with weak equivalences and duality; recall from Definition 2.1.1 that this implies in particular that $\mathbb{D} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ preserves congressions and weak equivalences.

Definition A.3.4. A complicial structure on an exact category with weak equivalences and duality $(\mathcal{E}, w, \mathbb{D}, \eta)$ is a biexact functor $\otimes : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$ which refines to a duality-preserving form functor of exact categories with duality, i.e. if there is a natural isomorphism rendering the diagram

$$\begin{array}{ccc} \mathscr{C}_{\mathbb{Z}}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} & \xrightarrow{[-,\mathbb{1}]\times\mathbb{D}} & \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \\ \downarrow{\scriptstyle \otimes^{\mathrm{op}}} & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle \otimes} \\ \mathcal{E}^{\mathrm{op}} & \xrightarrow{\quad\mathbb{D}\quad} & \mathcal{E} \end{array}$$

commutative. That this is the case for $\mathscr{C}_{\mathbb{Z}}$ with complicial structure induced by its symmetric monoidal structure, with duality $X \mapsto [X, \mathbb{1}]$, follows from the natural isomorphism

$$[X, A] \otimes [Y, B] \to [X \otimes Y, A \otimes B], \quad f \otimes g \mapsto (f \otimes g : x \otimes y \mapsto (-1)^{|x||g|} f(x) \otimes g(y)).$$

More generally, to check this for the canonical example of chain complexes over an exact category with weak equivalences and duality, note that the duality on $\mathscr{C}_{\mathbb{Z}}$ is extended from $\mathrm{Proj}_{\mathbb{Z}}^{\mathrm{op}} \to \mathrm{Proj}_{\mathbb{Z}}$, $A \mapsto A^\vee := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$, and that for $x \in \mathcal{E}$, there is a natural isomorphism

$$\mathbb{D}(A \otimes x) = \mathbb{D}(x^{\oplus \mathrm{rk}(A)}) \cong \mathbb{D}(x)^{\oplus \mathrm{rk}(A)} = \mathbb{D}(x)^{\oplus \mathrm{rk}(A^\vee)} \cong A^\vee \otimes \mathbb{D}(x).$$

It then follows that for $X \in \mathrm{Ch}_b(\mathcal{E})$ and $A \in \mathscr{C}_{\mathbb{Z}}$, we have isomorphisms $\mathbb{D}(A \otimes X) \cong [A, \mathbb{1}] \otimes \mathbb{D}(X)$ given in degree n by

$$\begin{aligned} \mathbb{D}(A \otimes X)_n &= \mathbb{D}(\bigoplus_i A_i \otimes X_{-n-i}) \cong \bigoplus_i \mathbb{D}(A_i \otimes X_{-n-i}) \\ &\cong \bigoplus_i A_i^\vee \otimes \mathbb{D}(X_{-n-i}) = \bigoplus_i [A, \mathbb{1}]_{-i} \otimes \mathbb{D}(X)_{i+n} \\ &= ([A, \mathbb{1}] \otimes \mathbb{D}(X))_n. \end{aligned}$$

That this commutes with the differentials follows from commutativity of the square

$$\begin{array}{ccc} \mathbb{D}(A_i \otimes X_{-n-i}) & \xrightarrow{\begin{pmatrix} \mathbb{D}(d^A_{i+1} \otimes \mathrm{id}_{X_{-n-i}}) \\ (-1)^i\, \mathbb{D}(\mathrm{id}_{A_i} \otimes d^X_{1-n-i}) \end{pmatrix}} & \mathbb{D}(A_{i+1} \otimes X_{-n-i}) \oplus \mathbb{D}(A_i \otimes X_{1-n-i}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ (A_i)^\vee \otimes \mathbb{D}(X_{-n-i}) & \xrightarrow[\begin{pmatrix} (d^A_{i+1})^\vee \otimes \mathrm{id}_{\mathbb{D}(X_{-n-i})} \\ (-1)^{-i}\, \mathrm{id}_{(A_i)^\vee} \otimes \mathbb{D}(d^X_{1-n-i}) \end{pmatrix}]{} & (A_{i+1})^\vee \otimes \mathbb{D}(X_{-n-i}) \oplus (A_i)^\vee \otimes \mathbb{D}(X_{1-n-i}) \end{array}$$

for each i.

Remark A.3.5. The opposite of a complicial exact category is complicial exact, with structure map furnished by the duality in $\mathscr{C}_{\mathbb{Z}}$:

$$\mathscr{C}_{\mathbb{Z}} \times \mathcal{E}^{\mathrm{op}} \xrightarrow{[-,\mathbb{1}]\times \mathrm{id}_{\mathcal{E}^{\mathrm{op}}}} \mathscr{C}_{\mathbb{Z}}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \xrightarrow{\otimes^{\mathrm{op}}} \mathcal{E}^{\mathrm{op}}. \tag{A.4}$$

Recollection A.3.6. For $\mathcal{E}$ any exact category, an object $z \in \mathcal{E}$ is said to be injective if the restriction $i^*\colon \mathrm{Hom}_{\mathcal{E}}(y, z) \to \mathrm{Hom}_{\mathcal{E}}(x, z)$ along each ingression $x \rightarrowtail y$ is a surjection; dually, z is projective if for any egression $p\colon y \twoheadrightarrow x$, the map $p_*\colon \mathrm{Hom}_{\mathcal{E}}(z, y) \to \mathrm{Hom}_{\mathcal{E}}(z, x)$ is a surjection. $\mathcal{E}$ is said to have enough injectives (projectives) if each $x \in \mathcal{E}$ admits an ingression $x \rightarrowtail I_x$ into an injective object (an egression $P_x \twoheadrightarrow x$ from a projective object). A **Frobenius exact category** is an exact category with enough injectives and projectives, and in which the classes of injectives and projectives coincide.

Recall [Sch11, Lem. A.2.16] that any complicial exact category has an underlying Frobenius exact category, as follows: writing as usual $C := 0 \to \mathbb{Z} = \mathbb{Z} \to 0$ in degrees 0 and 1, declare an ingression $i\colon x \rightarrowtail y$ to be Frobenius if for each $u \in \mathcal{E}$, the map $i^*\colon \mathrm{Hom}_{\mathcal{E}}(y, C \otimes u) \to \mathrm{Hom}_{\mathcal{E}}(x, C \otimes u)$ is surjective. Dually, an egression $p\colon y \twoheadrightarrow x$ is Frobenius if for each $u \in \mathcal{E}$, $p_*\colon \mathrm{Hom}_{\mathcal{E}}(C \otimes u, y) \to \mathrm{Hom}_{\mathcal{E}}(C \otimes u, x)$ is a surjection. The Frobenius ingressions and egressions equip $\mathcal{E}$ with a Frobenius exact structure, in which an object is injective-projective if it is a retract of $C \otimes u$ for some u, and we call such an injective-projective **(Frobenius) contractible**. We write $\mathcal{E}_{\mathrm{Frob}}$ for this Frobenius exact structure.

Definition A.3.7. Given a Frobenius exact category $\mathcal{E}$, call morphisms $f, g : x \to y$ Frobenius homotopic if their difference factors through an injective-projective object, and call a map $f : x \to y$ a Frobenius equivalence if it admits a two-sided inverse up to Frobenius homotopy. Frobenius homotopy then defines an equivalence relation on $\mathrm{Hom}_{\mathcal{E}}(x, y)$ for each $x, y \in \mathcal{E}$, and the corresponding quotient category is the so-called stable category $\underline{\mathcal{E}}$. This can be equipped with a canonical triangulated structure [Sch11, §2.14], as follows: choose for each x an ingression into an injective-projective $x \rightarrowtail I_x$; then a triangle in $\underline{\mathcal{E}}$ is distinguished if it is isomorphic to one of the form

$$x \xrightarrow{f} y \to I_x \cup_x y \to I_x/x.$$

This triangulation is easily shown to be that which arises from stability of the localisation $L_{\mathrm{Frob}}(\mathcal{E})$ at those maps sent to isomorphisms by the functor $\mathcal{E} \to \underline{\mathcal{E}}$.

Example A.3.8. The Frobenius exact structure on $\mathrm{Ch}_b(\mathcal{E})$ arising from its complicial exact structure comprises the degreewise-split monomorphisms and epimorphisms. A map of complexes is then a Frobenius equivalence if and only if it is a chain homotopy equivalence.

Remark A.3.9. Any complicial exact category $\mathcal{E}$ supports a notion of simplicial homotopy: call maps $f, g : x \to y$ simplicially homotopic if there exists a map $H : \Delta^1 x \to y$ rendering the diagram

$$\begin{array}{ccc} x & & \\ \downarrow{\scriptstyle d^1} & \searrow^{f} & \\ \Delta^1 x & \xrightarrow{H} & y \\ \uparrow{\scriptstyle d^0} & \nearrow_{g} & \\ x & & \end{array}$$

commutative; here we write Δ^1 for the image of the standard 1-simplex under the functor $N_* \mathbb{Z}[-] : s\mathcal{S}et \to \mathscr{C}_{\mathbb{Z}}$, and d^0, d^1 respectively for the maps induced by the inclusions $\binom{0}{1}, \binom{1}{0} : \mathbb{Z}[0] = \Delta^0 \to \Delta^1$. Unsurprisingly, maps $f, g : x \to y$ are simplicially homotopic if and only if they are Frobenius homotopic: a Frobenius homotopy $f - g : x \to C \otimes u \to y$, factors by injectivity of $C \otimes u$ over the Frobenius ingression $x \rightarrowtail C \otimes x$ induced by the map $\mathbb{1} \rightarrowtail C$. With reference to the retract diagram

$$\begin{array}{ccccccccc} C & & 0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle i} & & & & \| & & \downarrow{\scriptstyle \binom{-1}{1}} & & \\ \Delta^1 & & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\binom{-1}{1}} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle p} & & & & \| & & \downarrow{\scriptstyle (-1\ 0)} & & \\ C & & 0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

we then have a simplicial homotopy

$$\begin{array}{ccccc} x & & \xrightarrow{\quad g-f \quad} & & \\ {\scriptstyle d^1}\downarrow & & & & \\ \Delta^1 x & \xrightarrow{p} & Cx & \longrightarrow & y, \\ {\scriptstyle d^0}\uparrow & & & & \\ x & & \xrightarrow{\quad 0 \quad} & & \end{array}$$

and hence from f to g by the evident additivity of simplicial homotopy. Conversely, given a simplicial homotopy $f \sim g$, the difference $d^0 - d^1 : x \to \Delta^1 x$ factors through the inclusion $i : C \otimes x \to \Delta^1 x$, and we have a factorisation $g - f : x \xrightarrow{d^0 - d^1} C \otimes x \xrightarrow{Hi} y$.

Let $(\mathcal{E}, w)$ be an exact category with weak equivalences. An object $x \in \mathcal{E}$ is said to be (w-)acyclic if the unique map $0 \to x$ is a weak equivalence. By [Sch24b, Lem. 7.1], an ingression $x \rightarrowtail y$ resp. egression $y \twoheadrightarrow x$ is a weak equivalence if and only if its cokernel resp. kernel is acyclic (we call such ingressions or egressions trivial). Accordingly, the full subcategory $\mathcal{E}^w \subset \mathcal{E}$ of acyclic objects is closed under extensions in $\mathcal{E}$, and we equip $\mathcal{E}^w$ with the induced exact structure.

Write $\mathcal{E}_{\mathrm{in}}$ and $\mathcal{E}_{\mathrm{eg}}$ for the classes of ingressions and egressions in $\mathcal{E}$. Then the pairs $(w \cap \mathcal{E}_{\mathrm{in}}, \mathcal{E}_{\mathrm{eg}})$ and $(\mathcal{E}_{\mathrm{in}}, w \cap \mathcal{E}_{\mathrm{eg}})$ each form a functorial factorisation system on $\mathcal{E}$, in the following sense: given a map $f : x \to y$, we have a commutative diagram

$$\begin{array}{ccc} x & \overset{\binom{\iota_x}{f}}{\rightarrowtail} & Cx \oplus y \\ {\scriptstyle \binom{\mathrm{id}_x}{0}}\downarrow\wr & \searrow^{f} & \wr\downarrow{\scriptstyle (0\ \mathrm{id}_y)} \\ x \oplus Py & \underset{(0\ \pi_y)}{\twoheadrightarrow} & y, \end{array} \tag{A.5}$$

where we write $\iota_x : x \rightarrowtail Cx$ and $\pi_y : Py \twoheadrightarrow y$ for the maps induced respectively by $\mathbb{1} \rightarrowtail C$ and $P \twoheadrightarrow \mathbb{1}$ in $\mathscr{C}_{\mathbb{Z}}$. Note that each vertical map is a Frobenius equivalence by contractibility of the respective kernel or cokernel.

Remark A.3.10. For $(\mathcal{E}, w)$ complicial exact with weak equivalences, write w_{Frob} for the subcategory of Frobenius equivalences. We claim $w_{\mathrm{Frob}} \subset w$: for a map $f : x \to y$ in $\mathcal{E}$, define the mapping cone $C(f)$ as the pushout

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \iota_x}\downarrow & & \downarrow \\ Cx & \longrightarrow & C(f). \end{array} \tag{A.6}$$

From the diagram

$$\begin{array}{ccccc} & & Cx & & \\ & & \downarrow{\scriptstyle \binom{\mathrm{id}_x}{0}} & & \\ x & \overset{\binom{f}{-\iota_x}}{\rightarrowtail} & Cx \oplus y & \twoheadrightarrow & C(f) \\ & {\scriptstyle f}\searrow & \wr\downarrow & & \\ & & y & & \end{array}$$

in which horizontal and vertical sequences are exact, we see that f lies in w if and only if its cone lies in $\mathcal{E}^w$. The inclusion $w_{\mathrm{Frob}} \subset w$ then follows from the exact inclusion $\mathcal{E}^{w_{\mathrm{Frob}}} \subset \mathcal{E}^w$, which holds since the map $0 \to C \otimes u$ is the tensor product $(0 \to C) \otimes \mathrm{id}_u$ of weak equivalences, and $\mathcal{E}^w$ is closed under retracts.

The pair $(\mathcal{E}_{\mathrm{Frob}}, w)$ is an exact category with weak equivalences, since w is stable under pullback resp. pushout along egressions resp. ingressions, and in particular Frobenius egressions resp. ingressions. We accordingly have exact inclusions

$$(\mathcal{E}_{\mathrm{Frob}}, w_{\mathrm{Frob}}) \subset (\mathcal{E}_{\mathrm{Frob}}, w) \subset (\mathcal{E}, w)$$

of exact categories with weak equivalences, with underlying functor $\mathrm{id}_{\mathcal{E}}$.

Remark A.3.11. The inclusion $(\mathcal{E}_{\mathrm{Frob}}, w) \to (\mathcal{E}, w)$ induces an equivalence on Dwyer-Kan localisations $L_w(\mathcal{E}_{\mathrm{Frob}}) \simeq L_w(\mathcal{E})$ on underlying ∞-categories, by the universal property of localisation and the observation that $\mathcal{E}$ and $\mathcal{E}$ have the same underlying additive category.

Remark A.3.12. an extension-closed subcategory $\mathcal{U}$ of an exact ∞-category $\mathcal{E}$ (with the induced exact structure) is **left special** [SW25, Def. 1.1] if for each egression $x \twoheadrightarrow u$ with $u \in \mathcal{U}$, there is some map $v \to x$ with $v \in \mathcal{U}$ such that the composite $v \to u$ is an egression in $\mathcal{U}$. The inclusion $\mathcal{E}^w \subset \mathcal{E}$ is trivially left special, since given $x \twoheadrightarrow u$ with u acyclic, we may precompose with the Frobenius egression $Px \twoheadrightarrow x$, with Px Frobenius acyclic. It then follows [SW25, Th. 1.2] that the induced functor $D_b(\mathcal{E}^w) \to D_b(\mathcal{E})$ is fully faithful.

For $n \geq 0$, write $\Delta^n \in \mathscr{C}_{\mathbb{Z}}$ for image of the standard n-simplex under the composite $s\mathcal{S}et \xrightarrow{\mathbb{Z}[-]} s\mathcal{A}b \xrightarrow{N} \mathrm{Ch}_{\geq 0}(\mathbb{Z})$. There is a mapping space bifunctor

$$\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to s\mathcal{A}b, \quad (x, y) \mapsto \mathrm{Map}_{\Delta}(x, y) := \mathrm{Hom}_{\mathcal{E}}(\Delta^{\bullet} x, y),$$

equipping $\mathcal{E}$ with a simplicial enrichment such that the resulting simplicial category $\mathcal{E}_{\Delta}$ is fibrant in the Bergner model structure. By Remark A.3.9, we have $\pi_0 \mathrm{Map}_{\Delta}(x, y) \cong \mathrm{Hom}_{\underline{\mathcal{E}}}(x, y)$, and writing $L_{\mathrm{Frob}}(\mathcal{E})$ for the Dwyer-Kan localisation at the Frobenius equivalences, we have a factorisation

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\gamma} & L_{\mathrm{Frob}}(\mathcal{E}) \\ & \searrow & \downarrow f \\ & & N_{\Delta}(\mathcal{E}_{\Delta}), \end{array}$$

with f clearly essentially surjective, and the functor $\mathcal{E} \to N_{\Delta}(\mathcal{E}_{\Delta})$ induced by the inclusion of 0-simplices. Now it follows from Proposition 3.1.5 that $\mathrm{Map}_{\Delta}(-,-)$ sends Frobenius equivalences in each variable to homotopy equivalences. Applying Proposition 3.1.7 to the functor $\mathrm{Map}_{\Delta}(-, y)$ yields a natural equivalence

$$\mathrm{Map}_{\Delta}(x, y) \simeq \operatorname*{colim}_{J_{x,\mathrm{Frob}}^{\mathrm{op}}} \mathrm{Hom}(-, y),$$

where we write $J_{x,\mathrm{Frob}} \subset (\mathcal{E} \downarrow x)$ for the full subcategory spanned by the Frobenius egressions over x which lie in w_{Frob}, and the colimit is taken in the underlying ∞-category of simplicial abelian groups. Under this equivalence, the map $\mathrm{Hom}_{\mathcal{E}}(x, y) \to \mathrm{Map}_{\Delta}(x, y)$ identifies with the canonical inclusion $\mathrm{Hom}_{\mathcal{E}}(x, y) \to \operatorname*{colim}_{J_x^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{E}}(-, y)$, so that the functor f is fully faithful, and $N_{\Delta}(\mathcal{E}_{\Delta})$ is a model for the Dwyer-Kan localisation $L_{\mathrm{Frob}}(\mathcal{E})$.

Proposition A.3.13. *The Dwyer-Kan localisation of any complicial exact category with weak equivalences is a stable ∞-category. Moreover, an exact functor between such induces an exact functor of stable ∞-categories.*

Proof. This is in spirit the same as [BC20, Prop. 2.7], but we provide the details. We first note that $(\mathcal{E}, w)$ is an ∞-category of fibrant objects (Lemma 3.1.2), so that by [Cis19, Prop. 7.5.6], the localisation $L_w(\mathcal{E})$ admits finite limits, and the localisation $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ is left exact. The same argument applied to the complicial exact category with weak equivalences $(\mathcal{E}^{op}, w^{op})$ furnishes us with finite colimits, and since 0 is both initial and final, $L_w(\mathcal{E})$ is pointed.
Now each map in $L_w(\mathcal{E})$ is up to equivalence the image of an ingression under γ, and hence every cofibre sequence is up to equivalence the image under γ of a pushout

$$\begin{array}{ccc} x & \rightarrowtail & y \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ 0 & \rightarrowtail & z. \end{array}$$

Since $x \rightarrowtail y \twoheadrightarrow z$ is a congression, this is also a pullback and hence localises to a fibre sequence in $L_w(\mathcal{E})$. The same argument applied to $(\mathcal{E}^{op}, w^{op})$, yields that fibre and cofibre sequences coincide in $L_w(\mathcal{E})$, which is thus stable. For the last claim, an exact functor $(\mathcal{E}, w) \to (\mathcal{E}', w')$ between complicial exact categories with weak equivalences preserves zero objects and congressions by definition, and hence by [Cis19, Prop. 7.5.6] descends to an exact functor $L_w(\mathcal{E}) \to L_{w'}(\mathcal{E}')$. □

Remark A.3.14. By the universal property of the derived ∞-category [BCKW24, Cor. 7.4.12], the localisation $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ factors over the canonical inclusion $\mathcal{E} \hookrightarrow D_b(\mathcal{E})$. Write $\mathcal{E}^w \subset \mathcal{E}$ for the inclusion of the exact subcategory of w-acyclics, inducing a functor $D_b(\mathcal{E}^w) \to D_b(\mathcal{E})$ which is fully faithful by Remark A.3.12. We claim that the functor $D_b(\mathcal{E}) \to L_w(\mathcal{E})$ exhibits $L_w(\mathcal{E})$ as the Verdier quotient $D_b(\mathcal{E})/D_b(\mathcal{E}^w)$: for $\mathcal{C} \in \mathcal{C}at_\infty^{st}$, write $\mathrm{Fun}_w^{ex}(\mathcal{E}, \mathcal{C})$ for the full subcategory of exact functors inverting weak equivalences, and $\mathrm{Fun}^{ex}_{D_b(\mathcal{E}^w)}(D_b(\mathcal{E}), \mathcal{C})$ for the full subcategory of functors annihilating $D_b(\mathcal{E}^w)$. We claim that $\mathrm{Fun}_w^{ex}(\mathcal{E}, \mathcal{C})$ coincides with the image of $\mathrm{Fun}^{ex}_{D_b(\mathcal{E}^w)}(D_b(\mathcal{E}), \mathcal{E}) \subset \mathrm{Fun}^{ex}(D_b(\mathcal{E}), \mathcal{C})$ under the equivalence $\mathrm{Fun}^{ex}(D_b(\mathcal{E}), \mathcal{C}) \simeq \mathrm{Fun}^{ex}(\mathcal{E}, \mathcal{C})$: given a functor $F : \mathcal{E} \to \mathcal{C}$ inverting w, then since any w-acyclic x arises as the kernel of a trivial egression, say $(\,0\;1\,) : x \oplus x \to x$, the composite $\mathcal{E}^w \subset \mathcal{E} \to \mathcal{C}$ vanishes, and accordingly the induced functor $D_b(\mathcal{E}^w) \to \mathcal{C}$ also. Conversely, given a functor $G : D_b(\mathcal{E}) \to \mathcal{C}$ vanishing on $D_b(\mathcal{E}^w)$, the composite $\mathcal{E}^w \subset D_b(\mathcal{E}) \to \mathcal{C}$ vanishes, and since any weak equivalence in $\mathcal{E}$ can be written as the composite of a trivial ingression followed by a trivial egression, of which the cokernel and kernel vanish in $\mathcal{C}$, we see that $G|_{\mathcal{E}}$ inverts maps in w. Accordingly, the horizontal map

$$\begin{array}{ccc} \mathrm{Fun}^{ex}(L_w(\mathcal{E}), \mathcal{C}) & \longrightarrow & \mathrm{Fun}^{ex}{}_{D_b(\mathcal{E}^w)}(D_b(\mathcal{E}), \mathcal{C}) \\ & \searrow^{\sim} & \downarrow \wr \\ & & \mathrm{Fun}_w^{ex}(\mathcal{E}, \mathcal{C}) \end{array}$$

is an equivalence by 2-of-3, exhibiting $L_w(\mathcal{E}) \simeq D_b(\mathcal{E})/D_b(\mathcal{E}^w)$.

We record the following for completeness. We use the phrase **(small) $\mathbb{Z}$-linear stable ∞-category** to indicate a small stable ∞-category $\mathcal{C}$ which is a module over the symmetric monoidal category $\mathrm{Perf}_{\mathbb{Z}}^{\otimes}$ with symmetric monoidal structure induced by the tensor product of chain complexes, such that the tensor product $\otimes : \mathrm{Perf}_{\mathbb{Z}} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable.

Lemma A.3.15. *The localisation $L_w(\mathcal{E})$ of a complicial exact category with weak equivalences is a $\mathbb{Z}$-linear stable ∞-category.*

Remark A.3.16. Following Hinich [Hin16, Def. 2.1.1], call a map $f : (\mathcal{C}, v) \to (\mathcal{D}, w)$ of marked simplicial sets a marked cocartesian fibration if:

(i) $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, and $f : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration;

(ii) a cocartesian lift of a marked arrow in $\mathcal{D}$ is marked in $\mathcal{C}$;

(iii) for each $\alpha : d \to d'$ in $\mathcal{D}$, the induced functor $\alpha_! : \mathcal{C}_d \to \mathcal{C}_{d'}$ preserves marked edges;

(iv) if $\alpha : d \to d'$ is marked in $\mathcal{D}$, the corresponding functor $\alpha_! : (\mathcal{C}_d, v \cap \mathcal{C}_d) \to (\mathcal{C}_{d'}, w \cap \mathcal{C}_{d'})$ induces an equivalence on localisations.

Then by [Hin16, Prop. 2.1.4] the induced functor $L(f) : L_v(\mathcal{C}) \to L_w(\mathcal{D})$ is (equivalent to) a cocartesian fibration (in the terminology of *ibid.*, $L(f)$ is a cocartesian fibration in $\mathcal{C}at_\infty$, i.e. admits a factorisation gi in $s\mathcal{S}et$, for g a cocartesian fibration and i a categorical equivalence).

Proof of Lemma A.3.15. A complicial structure on $\mathcal{E}$ is precisely a left tensoring over the symmetric monoidal category $\mathscr{C}_{\mathbb{Z}}^{\otimes}$, such that the structure maps preserve weak equivalences in each variable. Accordingly, we have a cocartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$, and identifications $\mathscr{C}_{\mathbb{Z}} \simeq \mathcal{C}_a$ and $\mathcal{E} \simeq \mathcal{C}_m$, for $\mathcal{LM}^{\otimes}$ the left module operad of [HA, §4.2.1]. Equip $\mathscr{C}_{\mathbb{Z}}$ with the subcategory of weak equivalences given by the chain homotopy equivalences. We define $w^{\otimes} \subset \mathcal{C}^{\otimes}$ to be the subcategory of weak equivalences generated by maps in $w \subset \mathcal{E}$ and the chain homotopy equivalences of $\mathscr{C}_{\mathbb{Z}}$ under the equivalences $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n$, for $n \geq 0$ and $\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1 \rangle}$, and $\mathcal{C}^{\otimes}_{(\langle 1 \rangle, \varnothing)} \simeq \mathscr{C}_{\mathbb{Z}}^{\otimes}$, $\mathcal{C}_{(\langle 1 \rangle, \{1\})} \simeq \mathcal{E}$. We claim that the functor $(\mathcal{C}^{\otimes}, w^{\otimes}) \to (\mathcal{LM}^{\otimes}, (\mathcal{LM}^{\otimes})^{\simeq})$ is a marked cocartesian fibration. Indeed, (i) is definitional, (ii) follows since a cocartesian lift of an equivalence is an equivalence, (iii) follows from the fact that $\otimes : \mathscr{C}_{\mathbb{Z}} \times \mathcal{E} \to \mathcal{E}$ preserves weak equivalences in each variable, and (iv) follows since the functor $\alpha_!$ is an equivalence if α is. The conclusion then follows from [Hin16, Prop. 2.1.4], the identification $L(\mathscr{C}_{\mathbb{Z}}) \simeq \mathrm{Perf}_{\mathbb{Z}}$, and the observation that the induced functor $\otimes : \mathrm{Perf}_{\mathbb{Z}} \times L_w(\mathcal{E}) \to L_w(\mathcal{E})$ is biexact, which follows from Proposition A.3.13. ☐

Remark A.3.17. Restricting the above argument along the inclusion $\mathrm{Assoc}^{\otimes} \subset \mathcal{LM}^{\otimes}$ yields, as in [Hin16], the monoidal structure on $\mathrm{Perf}_{\mathbb{Z}}$ such that the localisation $\mathscr{C}_{\mathbb{Z}} \to \mathrm{Perf}_{\mathbb{Z}}$ refines to a monoidal functor of monoidal categories.

Lemma A.3.18. *Suppose $\mathcal{E}$ is a complicial exact category. The bifunctor* $\mathrm{Map}_{\Delta}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to s\mathcal{A}b$ *sends Frobenius congressions in each variable to homotopy fibre sequences.*

Proof. Let $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{E}$ be a Frobenius congression. There is a diagram of cosimplicial congressions

$$\begin{array}{ccccc} \tilde{x}^{\bullet} & \rightarrowtail & \tilde{y}^{\bullet} & \twoheadrightarrow & \tilde{z}^{\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^{\bullet}x & \rightarrowtail & \Delta^{\bullet}y & \twoheadrightarrow & \Delta^{\bullet}z \\ \downarrow & & \downarrow & & \downarrow \\ cx & \rightarrowtail & cy & \twoheadrightarrow & cz, \end{array} \tag{A.7}$$

where the vertical congressions are induced levelwise by

$$\tilde{\Delta}^n \rightarrowtail \Delta^n \twoheadrightarrow \Delta^0$$

in $\mathscr{C}_{\mathbb{Z}}$. Since $\tilde{x}^n, \tilde{y}^n, \tilde{z}^n$ are Frobenius contractible, the vertical congressions and the top row split. For any $v \in \mathcal{E}$, the induced diagram

$$\begin{array}{ccccc}
c\operatorname{Hom}_{\mathcal{E}}(z,v) & \rightarrowtail & c\operatorname{Hom}_{\mathcal{E}}(y,v) & \twoheadrightarrow & c\operatorname{Hom}_{\mathcal{E}}(x,v) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_{\mathcal{E}}(\Delta^{\bullet}z,v) & \rightarrowtail & \operatorname{Hom}_{\mathcal{E}}(\Delta^{\bullet}y,v) & \twoheadrightarrow & \operatorname{Hom}_{\mathcal{E}}(\Delta^{\bullet}x,v) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_{\mathcal{E}}(\tilde{z}^{\bullet},v) & \rightarrowtail & \operatorname{Hom}_{\mathcal{E}}(\tilde{y}^{\bullet},v) & \twoheadrightarrow & \operatorname{Hom}_{\mathcal{E}}(\tilde{x}^{\bullet},v),
\end{array}$$

has the lower row and each column a homotopy fibre sequence in simplicial abelian groups, since the projections are degreewise split surjective. The top row is clearly a homotopy fibre sequence, since maps of discrete simplicial sets are Kan fibrations, and hence so is the middle row. The dual argument works for $\operatorname{Map}_{\Delta}(v,-)$. □

Lemma A.3.19. *The space of sections of any trivial Frobenius egression is a contractible Kan complex.*

Proof. The space of such sections is nonempty since $\ker(p)$ is Frobenius contractible and hence injective-projective. Since p_* is split by s_* for any such section, we see from above that $p_* : \operatorname{Map}_{\Delta}(x,y) \to \operatorname{Map}_{\Delta}(x,x)$ is a degreewise surjective homotopy equivalence of simplicial abelian groups, and hence a trivial fibration in the classical model structure on $s\mathcal{A}b$. Accordingly, the pullback

$$\begin{array}{ccc}
\operatorname{Sec}(p) & \longrightarrow & \operatorname{Map}_{\Delta}(x,y) \\
\downarrow & & \wr\downarrow p_* \\
\Delta^0 & \xrightarrow{\mathrm{id}_x} & \operatorname{Map}_{\Delta}(x,x)
\end{array}$$

is homotopy cartesian, and the left vertical arrow is thus a trivial fibration. □

For $(\mathcal{E}, w)$ a complicial exact category with weak equivalences and $x \in \mathcal{E}$, write $J_x \subset I_x \subset (\mathcal{E} \downarrow x)$ for the subcategories of the slice category spanned by trivial egressions resp. weak equivalences over x.

Remark A.3.20. Recall from Lemma 3.1.2 that $\mathcal{E}$ is naturally an ∞-category of fibrant objects in the sense of [Cis19, §7.5]. For each $x \in \mathcal{E}$, the category $(\mathcal{E} \downarrow x)$ acquires the structure of an ∞-category with weak equivalences and fibrations by declaring a map over x to be a fibration resp. weak equivalence in $(\mathcal{E} \downarrow x)$ if its image under the canonical right fibration

$$(\mathcal{E} \downarrow x) \to \mathcal{E}$$

is. The subcategory of acyclic objects $(\mathcal{E} \downarrow x)^w$ then coincides with I_x, which inherits the structure of an ∞-category with weak equivalences and fibrations, in which a fibrant object is precisely an object of J_x. The inclusion $J_x \subset I_x$ is then final by [Cis19, Prop. 7.6.8].

Suppose $\mathcal{C} \in \mathcal{C}at_\infty$ is equipped with classes $v \subseteq w \subseteq \mathcal{C}$ of weak equivalences, so that by the universal property of localisation, there is an essentially unique functor $\gamma : L_v(\mathcal{C}) \to L_w(\mathcal{C})$ such that the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\gamma_w} & L_w(\mathcal{C}) \\
& {\scriptstyle\gamma_v}\searrow \quad \nearrow{\scriptstyle\gamma} & \\
& L_v(\mathcal{C}) &
\end{array}$$

commutes. Then γ is the localisation of $L_v(\mathcal{C})$ at the essential image of w under γ_v, by the following argument. For any ∞-category $\mathcal{D}$, the restriction $\gamma^* : \mathrm{Fun}(L_w(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(L_v(\mathcal{C}), \mathcal{D})$ is fully faithful, since the composite $\gamma_v^* \circ \gamma^*$ is. Clearly the essential image of γ^* lies in the full subcategory on those functors inverting $\gamma_v(w)$, and conversely for any functor $F : L_v(\mathcal{C}) \to \mathcal{D}$ inverting $\gamma_v(w)$, there is an essentially unique functor $\widetilde{F} : L_w(\mathcal{C}) \to \mathcal{D}$ and a natural equivalence $\widetilde{F} \circ \gamma_w \simeq \widetilde{F} \circ \gamma \circ \gamma_v \simeq F \circ \gamma_v$. Since γ_v^* is fully faithful, there is a natural equivalence $F \simeq \widetilde{F} \circ \gamma$.

We now apply the above to $w_{\mathrm{Frob}} \subseteq w \subseteq \mathcal{E}$. Writing $\gamma_{\mathrm{Frob}} : \mathcal{E} \to L_{\mathrm{Frob}}(\mathcal{E})$ for the Dwyer-Kan localisation at the Frobenius equivalences and $\gamma : \mathcal{E} \to L_w(\mathcal{E})$ for the localisation at w, we see that the induced functor $L_{\mathrm{Frob}}(\mathcal{E}) \to L_w(\mathcal{E})$ is a localisation at the essential image of w under γ_{Frob}. Since $f \in w$ if and only if $C(f) \in \mathcal{E}^w$, this is equivalently the Verdier quotient at the essential image under γ_{Frob} of the extension-closed subcategory of w-acyclics $\mathcal{E}^w \subset \mathcal{E}$. We note that the full subcategory $L_{\mathrm{Frob}}(\mathcal{E}^w) \subset L_{\mathrm{Frob}}(\mathcal{E})$ is closed under retracts: indeed, a retract diagram $\overline{x} \to \overline{y} \to \overline{x}$ with $\overline{y} \in L_{\mathrm{Frob}}(\mathcal{E}^w)$ can be lifted to a diagram

$$\begin{array}{ccc} x & \xrightarrow{\iota} & y \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \rho} \\ & & x \end{array}$$

with $y \in \mathcal{E}^w$ which commutes up to Frobenius homotopy, so that the difference $\mathrm{id}_x - \rho\iota$ factors over the map $\iota_x : x \rightarrowtail Cx$, say via $r : Cx \to x$. But then the diagram

$$\begin{array}{ccc} x & \xrightarrow{\binom{\iota}{\iota_x}} & y \oplus Cx \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle (\rho\ r)} \\ & & x \end{array}$$

commutes on the nose, exhibiting $x \in \mathcal{E}^w$, so that $\overline{x} \in L_{\mathrm{Frob}}(\mathcal{E}^w)$.

Remark A.3.21. The above observation also shows that the class of weak equivalences in a complicial exact category with weak equivalences is saturated: suppose $f : x \to y$ is a map in $\mathcal{E}$ that is inverted in $L_w(\mathcal{E})$. We use that a map is a Frobenius resp. w-equivalence precisely when its cone lies in $\mathcal{E}^{w_{\mathrm{Frob}}}$ resp. $\mathcal{E}^w$. If $\gamma_{\mathrm{Frob}}(f)$ is an equivalence in $L_{\mathrm{Frob}}(\mathcal{E})$, the cone $C(f)$ of f becomes trivial in $L_{\mathrm{Frob}}(\mathcal{E})$, and accordingly is a retract of some Frobenius contractible object in $\mathcal{E}$ and is Frobenius contractible. Otherwise, since $\gamma_w(f)$ becomes an equivalence in $L_w(\mathcal{E})$, its cofibre in $L_{\mathrm{Frob}}(\mathcal{E})$ (the image of $C(f)$ under γ_{Frob}) lies in the retract-closure of the essential image of $\mathcal{E}^w \subset \mathcal{E}$ under γ_{Frob} by the characterisation of kernels of Verdier projections, so that $C(f)$ lies in $\mathcal{E}^w$ by above, and f is in w.

Write $\pi : L_{\mathrm{Frob}}(\mathcal{E}) \to L_w(\mathcal{E})$ for the Verdier projection at the w-acyclics.

Proposition A.3.22. *Given a functor* $F : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$, *for* $\mathcal{E} = (\mathcal{E}, w)$ *complicial exact with weak equivalences, the right derived functor* $\mathbf{R}F : L_w(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ *at* w *satisfies*

$$\mathbf{R}F(x) \simeq \operatorname*{colim}_{J_x^{\mathrm{op}}} F.$$

Proof. By definition, $\mathbf{R}F$ is the left Kan extension of $\mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ along the localisation $\mathcal{E} \to L_w(\mathcal{E})$. The claim follows from [Cis19, Cor. 7.2.9, 7.2.18] as soon as we demonstrate that $\{J_x\}_{x \in \mathcal{E}}$ is a class of trivial fibrations in the sense of [Cis19, Def. 7.2.20]: indeed, then $J_x \subset I_x$ is a right calculus of fractions at x, and we may compute the left Kan extension as

$$\mathbf{R}F(x) \simeq \operatorname*{colim}_{J_x^{op}} F.$$

The claim follows from Brown's lemma ([Cis19, Cor. 7.4.13]), since trivial egressions are clearly closed under composition and pullback. □

Recollection A.3.23. Suppose $\mathcal{C}$ is a small category, and $F : \mathcal{C} \to \mathcal{C}at$ is a pseudofunctor. The **Grothendieck construction** on (or **unstraightening** of) F is the category $\int_{\mathcal{C}} F$, with objects pairs $(x \in \mathcal{C}, a \in F(x))$, and maps

$$(x, a) \xrightarrow{(f,\gamma)} (y, b)$$

the data of arrows $f : x \to y$, and $\gamma : F(f)(a) \to b$ in $F(y)$. The composition

$$(x, a) \xrightarrow{(f,\gamma)} (y, b) \xrightarrow{(g,\delta)} (z, c)$$

is $(gf, \delta \circ F(g)(\gamma)) : (x, a) \to (z, c)$. There is a tautological functor $\int_{\mathcal{C}} F \to \mathcal{C}$, $(x, a) \mapsto x$, which on nerves is a right fibration.

Proposition A.3.24. *Suppose given a small complicial exact category with weak equivalences* $(\mathcal{E}, w)$*, with Frobenius localisation* $\gamma : \mathcal{E} \to L_{Frob}(\mathcal{E})$*, and some presheaf of spaces* $F : \mathcal{E}^{op} \to \mathcal{S}$*. Then there is a weak equivalence of spaces*

$$\operatorname*{colim}_{w\mathcal{E}^{op}} F \to \operatorname*{colim}_{w\mathcal{E}^{op}} \gamma^* \mathbf{R}F.$$

Proof. This follows from the Fubini theorem for unstraightening: given a functor $F : \mathcal{C} \to \mathcal{C}at_\infty$ classifying a cocartesian fibration $\int_{\mathcal{C}} F \to \mathcal{C}$, and a functor $G : \int_{\mathcal{C}} F \to \mathcal{S}$, there is an equivalence

$$\operatorname*{colim}_{\int_{\mathcal{C}} F} G \to \operatorname*{colim}_{x \in \mathcal{C}} \operatorname*{colim}_{F(x)} G_x, \tag{A.8}$$

where G_x is the restriction of G to the fibre $F(x)$; see for instance [Du23, Th. 4.4] (for the reader's convenience, and to correct a typo in *loc. cit.*, we reproduce the proof below). Consider now the functor

$$J_\bullet^{op} : w\mathcal{E}^{op} \to \mathcal{C}at \to \mathcal{C}at_\infty, \quad x \mapsto J_x^{op},$$

where a map $f : x \to y$ induces a functor $J_y^{op} \to J_x^{op}$ given by pullback along f. The Grothendieck construction on this functor has objects pairs $(x, y \overset{\sim}{\twoheadrightarrow} x)$, and maps

$$(x, y \overset{\sim}{\twoheadrightarrow} x) \to (x', y' \overset{\sim}{\twoheadrightarrow} x')$$

given by a pair (f, γ), where $f : x' \xrightarrow{\sim} x$ is a weak equivalence in $\mathcal{E}$, and $\gamma : y' \to x' \times_x y$ is a map over x' in $\mathcal{E}$; by the universal property of the pullback, this is the data of a commutative diagram

$$\begin{array}{ccc} y' & \overset{\sim}{\twoheadrightarrow} & x'. \\ \downarrow \wr & & \downarrow \wr \\ y & \overset{\sim}{\twoheadrightarrow} & x \end{array}$$

The category $\int_{w\mathcal{E}^{op}} J_\bullet^{op}$ thus identifies with the full subcategory of $\mathrm{Ar}(w\mathcal{E})^{op}$ spanned by the trivial egressions. The degeneracy $s_0 : w\mathcal{E} \subset \mathrm{Ar}(w\mathcal{E}), x \mapsto (x = x)$ admits a right adjoint given by the source functor d_1, and this restricts upon taking opposites to an adjoint pair

$$w\mathcal{E}^{op} \underset{d_1^{op}}{\overset{s_0^{op}}{\rightleftarrows}} \int_{w\mathcal{E}^{op}} J_\bullet^{op},$$

so in particular the functor $w\mathcal{E}^{op} \subset \int_{w\mathcal{E}^{op}} J_\bullet^{op}$ is cofinal. The equivalence (A.8) and Proposition A.3.22 then give a natural equivalence

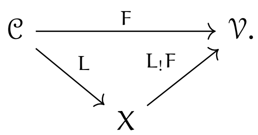
$$\operatorname*{colim}_{w\mathcal{E}^{op}} F \simeq \operatorname*{colim}_{\int_{w\mathcal{E}^{op}} J_\bullet^{op}} F \simeq \operatorname*{colim}_{x \in w\mathcal{E}^{op}} \operatorname*{colim}_{J_x^{op}} F = \operatorname*{colim}_{w\mathcal{E}^{op}} \gamma^* \mathbf{R}F,$$

and chasing through the various identifications we see that this is induced by the canonical map $F \Rightarrow \mathbf{R}F$. □

Lemma A.3.25. *Suppose* K *is a simplicial set,* $\varphi : K \to \mathcal{C}at_\infty$, $i \mapsto \mathcal{C}(i)$ *a diagram with cocartesian unstraightening* $p : \mathcal{C} \to K$, *and that* $F : \mathcal{C} \to \mathcal{V}$ *is a functor, for* $\mathcal{V}$ *some cocomplete ∞-category. Write* $\varphi|_{\mathcal{C}(i)}$ *for the composite* $\mathcal{C}(i) \subset \mathcal{C} \xrightarrow{\varphi} \mathcal{V}$. *Then there is a canonical equivalence*

$$\operatorname*{colim}_{i \in K} \operatorname*{colim}_{\mathcal{C}(i)} F|_{\mathcal{C}(i)} \xrightarrow{\sim} \operatorname*{colim}_{\mathcal{C}} F.$$

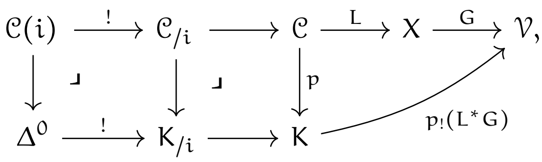
Proof. We reproduce the proof of Du in [Du23, Th. 4.4]. By [HTT, Cor. 3.3.4.3], there is an equivalence of ∞-categories $X := \mathrm{colim}_K\, \varphi \simeq \mathcal{C}[p\text{−cocart}^{-1}]$, with target the localisation of $\mathcal{C}$ at the p-cocartesian edges; if φ takes values in $\mathcal{S} \subset \mathcal{C}at_\infty$, we moreover have an equivalence of spaces $X \simeq |\mathcal{C}|$ by [HTT, Cor. 3.3.4.6]. Write $L : \mathcal{C} \to X$ for the localisation at the p-cocartesian edges, and consider the diagram



Since left Kan extensions compose and $L : \mathcal{C} \to X$ is cofinal as a localisation [Cis19, Prop7̇.1.10], we have

$$\mathrm{colim}_{\mathcal{C}}\, F \simeq \mathrm{colim}_X\, L_!F \simeq \mathrm{colim}_{\mathcal{C}}\, L^*L_!F,$$

so in computing the colimit of F we may suppose that F is restricted from functor $G : X \to \mathcal{S}$. Now consider for $i \in K$



where the vertical arrows are cocartesian fibrations, and the marked arrows cofinal. Then from the pointwise formula for left Kan extensions, for $i \in K$ we have

$$p_!(L^*G)(i) = \mathrm{colim}_{\mathcal{C}_{/i}}(G \circ L)(i) \simeq \mathrm{colim}_{\mathcal{C}(i)}(G \circ L)\,|_{\mathcal{C}(i)},$$

and

$$\begin{aligned} \mathrm{colim}_{i\in K}\, \mathrm{colim}_{\mathcal{C}(i)}\, F\,|_{\mathcal{C}(i)} &\simeq \mathrm{colim}_{i\in K}\, \mathrm{colim}_{\mathcal{C}(i)}(G \circ L)\,|_{\mathcal{C}(i)} \\ &\simeq \mathrm{colim}_{i\in K}\, p_!(L^*G) \simeq \mathrm{colim}_{\mathcal{C}}\, L^*G \simeq \mathrm{colim}_X\, G \simeq \mathrm{colim}_{\mathcal{C}}\, F. \end{aligned}$$

□

A.4 The underlying ∞-category of a model category

Write $\mathcal{C}at_\Delta$ for the category of small simplicially enriched categories and enriched functors. $\mathcal{C}at_\Delta$ admits a right-proper cofibrantly generated model structure [Ber07], the *Bergner model structure*, in which the weak equivalences are the Dwyer-Kan equivalences, i.e. those simplicial functors $f : \mathcal{C} \to \mathcal{D}$ inducing a weak equivalence on mapping spaces and an equivalence $\pi_0(f) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ on homotopy categories. Write $s\mathcal{S}et_{\mathrm{Joyal}}$ for the Joyal model structure on simplicial sets, in which the fibrant objects are the quasi-categories, i.e. those simplicial sets having the right lifting property with respect to all inner horn inclusions $\Lambda^n_i \subset \Delta^n$ for each $n \geq 2$. There is a Quillen equivalence [HTT, Th. 2.2.5.1]

$$\mathfrak{C}[-] : s\mathcal{S}et_{\mathrm{Joyal}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}at_\Delta : N_\Delta,$$

where the right Quillen adjoint N_Δ is the homotopy-coherent nerve. Bergner-fibrant simplicially enriched categories are precisely those enriched over Kan complexes, and accordingly for a $\mathcal{K}$an-enriched category $\mathcal{C}$, $N_\Delta(\mathcal{C})$ is a quasi-category.

Example A.4.1. For $\mathcal{M}$ a simplicial model category [Hir03, §9], the subcategory $\mathcal{M}^\circ \subset \mathcal{M}$ of fibrant-cofibrant objects is $\mathcal{K}$an-enriched; the homotopy-coherent nerve $N_\Delta(\mathcal{M}^\circ)$ is the *underlying ∞-category* of $\mathcal{M}$. The underlying ∞-category of a general model category $\mathcal{C}$ is given by the (derived) homotopy-coherent nerve of the simplicial localisation at the weak equivalences $N_\Delta(L^H(\mathcal{C}, w)^{\mathrm{fib}})$. For details, we refer the reader to [Hin16, §1.3].

There is a notion of homotopy (co)limits in a general simplicially enriched category $\mathcal{C}$ which generalises the corresponding notion for a diagram with values in a simplicial model category: given an enriched functor $F : \mathcal{I} \to \mathcal{A}$ with $\mathcal{A}$ Bergner-fibrant, a pair (x, η) consisting of $x \in \mathcal{A}$ and $\eta : F \Rightarrow \underline{x}$ a cone under F is said to be a *homotopy colimit* for F if for each $y \in \mathcal{A}$, the maps

$$\mathrm{Map}_{\mathcal{A}}(x, y) \to \lim_{i \in I} \mathrm{Map}_{\mathcal{A}}(F(i), y)$$

exhibit $\mathrm{Map}_{\mathcal{A}}(x, y)$ as a homotopy limit of the diagram

$$\mathrm{Map}_{\mathcal{A}}(F(-), y) : \mathcal{I} \to s\mathcal{S}et$$

with respect to the Kan-Quillen model structure on sSet, and dually for homotopy limits in $\mathcal{A}$ (see [HTT, §A.3.3]). The following theorem translates between this classical notion of homotopy (co)limits, and ∞-(co)limits of the diagrams obtained passing to homotopy-coherent nerves.

Theorem A.4.2 ([HTT, Th. 4.2.4.1])**.** *Suppose given a simplicial functor $F : \mathcal{I} \to \mathcal{C}$ of Bergner-fibrant simplicially enriched categories. Then for an object $c \in \mathcal{C}$ and a compatible family of maps $\{\eta_i : F(i) \to c\}_{i \in \mathcal{I}}$, the following are equivalent:*

(i) The maps η_i exhibit c as a homotopy colimit of F;

(ii) Write $f : N_\Delta(\mathcal{I}) \to N_\Delta(\mathcal{C})$ be the image of F under the homotopy-coherent nerve, and write $\overline{f} : N_\Delta(\mathcal{I})^\triangleright$ be the extension determined by the η_i. Then f is a colimit diagram in $N_\Delta(\mathcal{C})$.

Example A.4.3. Consider the subcategory $\mathcal{K}\mathrm{an} \subset \mathrm{s}\mathcal{S}\mathrm{et}$ spanned by the fibrant-cofibrant objects with respect to the Kan-Quillen simplicial model structure on simplicial sets. The homotopy-coherent nerve $\mathrm{N}_\Delta(\mathcal{K}\mathrm{an}) =: \mathcal{S}$ is the ∞-category of spaces, (co)limits in which correspond to homotopy (co)limits with respect to this model structure. In particular, pullbacks of spaces can be computed as 1-categorical pullbacks along fibrations.

B Polynomial functors

In this section we briefly review the Goodwillie-Lurie calculus of functors, and the 1-categorical analogue of polynomial functors on additive categories. Of particular importance is the bridge between these notions: an n-polynomial functor with valued in abelian groups on an additive category admits an essentially unique n-excisive extension valued in spectra on its stabilisation. The papers [BGMN22, §2], [BM24, §3], [CDH+I, §4.2] contain further details on n-excisive extensions.

B.1 Polynomial and n-excisive functors

Definition B.1.1. Let $\mathcal{A}, \mathcal{B}$ be additive ∞-categories with $\mathcal{B}$ weakly idempotent complete. A functor $F : \mathcal{A} \to \mathcal{B}$ is polynomial of degree ≤ 0 if it is a constant functor. For $d \geq 1$, F is **polynomial of degree** $\leq d$ if for each $x \in \mathcal{A}$, the functor

$$D_x F : \mathcal{A} \to \mathcal{B}, \quad y \mapsto \ker(F(x \oplus y) \to F(y))$$

is polynomial of degree $\leq d - 1$. Note that the kernel exists in this case, since $F(x \oplus y) \to F(y)$ admits a section induced by the inclusion $y \to x \oplus y$. For general $\mathcal{B}$ with weak idempotent completion $\mathcal{B} \to \mathcal{B}^\flat$, a functor $F : \mathcal{A} \to \mathcal{B}$ is polynomial of degree $\leq d$ if the composite $\mathcal{A} \xrightarrow{F} \mathcal{B} \to \mathcal{B}^\flat$ is. Denote by $\mathrm{Fun}^{n-\mathrm{poly}}(\mathcal{A}, \mathcal{B})$ for the full subcategory spanned by functors which are polynomial of degree $\leq n$, and $\mathrm{Fun}_*^{n-\mathrm{poly}}(\mathcal{A}, \mathcal{B})$ of such functor which are reduced, i.e. preserve zero objects.

Call a functor $B : \mathcal{A}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$ symmetric bilinear if B preserves direct sums in either variable, and there is a natural isomorphism $\sigma_{x,y} : B(x, y) \cong B(y, x)$ for each $x, y \in \mathcal{A}$.

Lemma B.1.2 ([Sch21, Lem. A.10])**.** *Given an additive category $\mathcal{A}$, a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$ is reduced and polynomial of degree ≤ 2 (quadratic) if and only if there is a symmetric bilinear functor $B : \mathcal{A}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$, and natural C_2-equivariant diagrams of abelian groups*

$$B(x, x) \xrightarrow{\tau_x} F(x) \xrightarrow{\rho_x} B(x, x),$$

such that $\rho_x \tau_x = 1 + \sigma_{x,x}$, and for parallel maps $f, g : x \to y$ and $\xi \in F(y)$, we have

$$(f + g)^\bullet(\xi) = f^\bullet(\xi) + g^\bullet(\xi) + \tau_x(B(f, g)(\rho_x(\xi))) \in F(x),$$

where we write $f^\bullet := F(f)$ etc.

Remark B.1.3. For quadratic F, we call the functor $B = B_F$ above the **polarisation** of F.

Definition B.1.4. Write $\mathcal{P}(n)$ for the power set of $[n] = \{0 < 1 < \cdots < n\}$, considered as a poset under inclusion. An $(n + 1)$-cube in an ∞-category $\mathcal{C}$ is a functor $f : N\mathcal{P}(n) \to \mathcal{C}$, and such a cube is said to be (co)cartesian if it is a limit (colimit) diagram, and strongly cocartesian if it is the left Kan extension of its restriction to $N\mathcal{P}_{\leq 1}(n)$, for $\mathcal{P}_{\leq 1}(n)$ the subposet on subsets of cardinality at most 1 (equivalently, any 2-face of f is a pushout).

Definition B.1.5. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories admitting finite colimits and limits, respectively. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be n**-excisive** if for any strongly cocartesian $(n+1)$-cube $\mathcal{P}(n) \to \mathcal{C}$, the cube

$$\mathcal{P}(n) \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is cartesian in $\mathcal{D}$. In the case $\mathcal{C}, \mathcal{D}$ are pointed, a 2-excisive functor $f : \mathcal{C} \to \mathcal{D}$ is **quadratic** if it is reduced, i.e. preserves zero objects.

Example B.1.6. For $\mathcal{C}, \mathcal{D}$ stable, a reduced functor $\mathcal{C} \to \mathcal{D}$ is 1-excisive precisely when it is exact, i.e. preserves finite limits and colimits.

Remark B.1.7. For $m \geq 0$, write $\iota_m : \mathbb{\Delta}^{op}|_{\leq m} \subset \mathbb{\Delta}^{op}$ for the inclusion of the full subcategory of $\mathbb{\Delta}^{op}$ spanned by ordinals $n \leq m$. A simplicial object $X : \mathbb{\Delta}^{op} \to \mathcal{C}$ is m-truncated if X is the left Kan-extension of its restriction $\iota_m^* X : \mathbb{\Delta}^{op}|_{\leq m} \to \mathcal{C}$, i.e. if the counit map $(\iota_m)_! \iota_m^* X \to X$ is an equivalence of functors $\mathbb{\Delta}^{op} \to \mathcal{C}$; X is said to be finite if it is n-truncated for some n, and the colimit of such an X is said to be a finite geometric realisations. Then by [BGMN22, Prop. 2.15] a functor $f : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is n-excisive if and only if the induced functor $\mathrm{Ho}(f) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is polynomial of degree $\leq n$ as a functor of additive categories, and f preserves finite geometric realisations.

B.2 Extending n-polynomial functors on additive categories

Write $\mathcal{A}dd_\infty$ resp. $\mathcal{C}at_\infty^{st}$ for the large ∞-categories of small additive resp. stable ∞-categories and additive resp. exact functors. The forgetful functor $\mathcal{C}at_\infty^{st} \to \mathcal{A}dd_\infty$ sending a stable category to its underlying additive category admits a left adjoint

$$\mathrm{St}^{add} : \mathcal{A}dd_\infty \to \mathcal{C}at_\infty^{st},$$

given by the restriction of Klemenc's stable envelope, and moreover restriction along the unit $\mathcal{A} \to \mathrm{St}(\mathcal{A})$ induces an equivalence of functor categories

$$\mathrm{Fun}^{1-exc}(\mathrm{St}^{add}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{Fun}^{add}(\mathcal{A}, \mathcal{E}),$$

for any stable ∞-category $\mathcal{E}$, where we write Fun^{add} for the full subcategory of additive functors; see [BCKW24, Th. 7.4.9]. Restricted to the full subcategory $\mathcal{A}dd_1 \subset \mathcal{A}dd_\infty$ of additive 1-categories and additive functors, there is a natural equivalence $\mathrm{St}^{add}(-) \simeq K_b(-)$, for $K_b(\mathcal{A})$ the localisation of the category of bounded chain complexes at the chain homotopy equivalences, a stable ∞-category by for instance [BC20, Prop. 2.7]. The following polynomial extension of this holds (see [BGMN22, §2] or [CDH+I, Prop. 4.2.18]).

Theorem B.2.1 ([BGMN22, Th. 2.19])**.** *For $\mathcal{A}$ an additive ∞-category with stabilisation $\iota_{\mathcal{A}} : \mathcal{A} \to \mathrm{St}^{add}(\mathcal{A})$ and $\mathcal{C}$ a stable ∞-category, there is an equivalence of ∞-categories*

$$\iota_{\mathcal{A}}^* : \mathrm{Fun}^{n-exc}(\mathrm{St}^{add}(\mathcal{A}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{n-poly}(\mathcal{A}, \mathcal{C})$$

B.3 n-excisive approximation

Suppose $\mathcal{C}, \mathcal{D}$ are ∞-categories, that $\mathcal{C}$ admits finite colimits, and that $\mathcal{D}$ is differentiable in the sense of Lurie [HA, Def. 6.1.1.6], i.e. admits finite limits and sequential colimits which commute (the functor colim : $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}_{\geq 0}), \mathcal{D}) \to \mathcal{D}$) is left exact). Then the inclusion $\mathrm{Fun}^{n-\mathrm{exc}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left-exact left adjoint P_n, the n-excisive approximation, by [HA, Th. 6.1.1.10].

Example B.3.1. For $\mathcal{C}$ and $\mathcal{D}$ as above and $F : \mathcal{C} \to \mathcal{D}$ a reduced functor, the 1-excisive approximation of F is given by

$$P_1 F := \varinjlim_{n} (\Omega^n \circ F \circ \Sigma^n) : \mathcal{C} \to \mathcal{D}.$$

If $\mathcal{C}, \mathcal{D}$ are stable ∞-categories and $\mathcal{D}$ is moreover differentiable (by [HA, Ex. 6.1.1.7] this is equivalent to $\mathcal{D}$ admitting countable coproducts), [CDH+I, Cons. 1.1.26] gives a formula for the 2-excisive approximation of a reduced functor $\mathcal{R} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Example B.3.2. For each $x \in \mathcal{C}$, the fibre sequence $\Omega x \to * \to x$ induces a nullcomposite sequence

$$\mathcal{R}(x) \to \Omega\mathcal{R}(\Omega x) \to \Omega B_{\mathcal{R}}(\Omega x, \Omega x)$$

which in the case $\mathcal{R}$ is 2-excisive is a fibre sequence by [CDH+I, Lem. 1.1.19]. In general, there is a natural map

$$\mathcal{R}(x) \to \Omega\, \mathrm{fib}(\mathcal{R}(\Omega x) \to B_{\mathcal{R}}(\Omega x, \Omega x)),$$

and the assignment $x \mapsto \mathrm{fib}(\mathcal{R}(\Omega x) \to B_{\mathcal{R}}(\Omega x, \Omega x))$ assembles into a functor $T_2\mathcal{R} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, admitting a natural transformation $\mathcal{R} \to T_2\mathcal{R}$. Iterating this, one obtains a model for the 2-excisive approximation as the sequential colimit

$$P_2\mathcal{R}(x) = \mathrm{colim}(\mathcal{R}(x) \to T_2\mathcal{R}(x) \to T_2^2\mathcal{R}(x) \to \dots).$$

B.4 Quadratic functors on exact ∞-categories

For $\mathcal{E}$ an exact ∞-category, write $\mathcal{E}_{\oplus} := (U\mathcal{E})^{\mathrm{split}} \to \mathcal{E}$ for the counit of the adjunction $(-)^{\mathrm{split}} \dashv U$ of Remark A.1.8, inducing on stable envelopes a functor $\mathrm{St}^{\mathrm{add}}(\mathcal{E}) \to \mathrm{St}(\mathcal{E})$ coinciding with the Verdier quotient by the subcategory of acyclics. Recall that restriction along the inclusion $\mathcal{E} \to \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ induces an equivalence

$$\mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}^{\mathrm{add}}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \to \mathrm{Fun}^{2-\mathrm{poly}}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}\mathrm{p}).$$

Call a functor $Q : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ resp. $Q : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}_{\geq 0}$ **quadratic left exact** if its polarisation $B_Q : \mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$, $B_Q(x, y) := \mathrm{fib}(Q(x \oplus y) \to Q(x) \oplus Q(y))$ is left exact in each variable, and if for each congression $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{E}$, the map $Q(z) \to Q(y)$ exhibits the source as the total fibre in spectra resp. connective spectra of the square

$$\begin{array}{ccc} Q(y) & \longrightarrow & Q(x) \\ \downarrow & & \downarrow \\ B_Q(y, x) & \longrightarrow & B_Q(x, x). \end{array}$$

Restriction along the functors $\mathcal{E} \to \mathrm{St}^{\mathrm{add}}(\mathcal{E}) \to \mathrm{St}(\mathcal{E})$ induces a triangle of fully faithful functors

$$\begin{array}{ccc} \mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}p) & \xrightarrow{\pi^*} & \mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}^{\mathrm{add}}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}p) \\ & \searrow & \downarrow \wr \\ & & \mathrm{Fun}^{2-\mathrm{poly}}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}p). \end{array}$$

Lemma B.4.1. *The essential image of the restriction* $\mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}p) \to \mathrm{Fun}^{2-\mathrm{poly}}(\mathcal{E}^{\mathrm{op}}, \mathcal{S}p)$ *is the full subcategory of quadratic left exact functors.*

Proof. Since the Yoneda embedding $\mathcal{E} \to \mathrm{St}(\mathcal{E})$ is exact, the restriction of a 2-excisive functor $\Qoppa : \mathrm{St}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}p$ to $\mathcal{E}$ exhibits the desired behaviour on congressions by [CDH+I, Cor. 1.1.21]. For the converse, we wish to show that a quadratic functor $Q : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}p$ with unique extension $\Qoppa : \mathrm{St}^{\mathrm{add}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}p$ vanishes on the subcategory of acyclics. Since this subcategory is by definition the smallest stable subcategory containing the elementary acyclics [Kle22, Def. 3.14], it suffices to show that $\Qoppa$ vanishes on the latter. Write B_Q for the polarisation of Q, noting that since $\mathcal{E} \subset \mathrm{St}(\mathcal{E})$ preserves direct sums, the identification $\Qoppa|_{\mathcal{E}^{\mathrm{op}}} \simeq Q$ furnishes an identification $B_\Qoppa|_{\mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}}} \simeq B_Q$. Write $j : \mathcal{E} \to \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ for the (additive) Yoneda embedding. For $x \overset{i}{\rightarrowtail} y \overset{p}{\twoheadrightarrow} z$ a congression in $\mathcal{E}$, write $C(i) := \mathrm{cofib}(j(i))$, and $E(i) := \mathrm{cofib}(C(i) \to j(z))$ for the associated elementary acyclic. For $w \in \mathcal{E}$, the restrictions $B_\Qoppa(w, -), B_\Qoppa(-, w) : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}p$ send congressions to fibre sequences, and accordingly the unique extensions $B_\Qoppa(w, -), B_\Qoppa(-, w) : \mathrm{St}^{\mathrm{add}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}p$ send the map $C \to j(z)$ to an equivalence. Since $\Qoppa$ is 2-excisive and coincides upon restriction $\mathcal{E}$ with Q, each of the maps $\Qoppa(j(z)) \to \Qoppa(j(y))$ and $\Qoppa(C) \to \Qoppa(j(y))$ exhibit the sources as the total fibre of the square

$$\begin{array}{ccc} \Qoppa((j(y)) & \longrightarrow & \Qoppa(j(x)) \\ \downarrow & & \downarrow \\ B_\Qoppa(j(y), j(x)) & \longrightarrow & B_\Qoppa(j(x), j(x)), \end{array} \tag{B.1}$$

and accordingly the map $\Qoppa(j(z)) \to \Qoppa(C)$ is an equivalence. Again by 2-excisivity the map $\Qoppa(E(i)) \to \Qoppa(j(z))$ also exhibits the source as the total fibre of the square

$$\begin{array}{ccc} \Qoppa(j(z)) & \longrightarrow & \Qoppa(C) \\ \downarrow & & \downarrow \\ B_\Qoppa(j(z), C) & \longrightarrow & B_\Qoppa(C, C). \end{array}$$

By bilinearity of $B_\Qoppa$, the fibre of the lower horizontal map identifies with the total fibre of the square

$$\begin{array}{ccc} B_\Qoppa(j(z), j(y)) & \longrightarrow & B_\Qoppa(j(z), j(x)) \\ \downarrow \simeq & & \downarrow \simeq \\ B_\Qoppa(C, j(y)) & \longrightarrow & B_\Qoppa(C, j(x)), \end{array}$$

which vanishes, and we see that $\Qoppa(E(i))$ vanishes since both horizontal arrows of B.1 are equivalences. □

Remark B.4.2. Suppose given an additive ∞-category $\mathcal{A}$, and a 2-polynomial functor $Q; \mathcal{A}^{\mathrm{op}} \to \mathcal{S}p_{\geq 0}$. Since the inclusion $i : \mathcal{S}p_{\geq 0} \subset \mathcal{S}p$ preserves fibres of split inclusions, the functor $iQ : \mathcal{A}^{\mathrm{op}} \to \mathcal{S}p$ is 2-polynomial, and extends uniquely to $\Qoppa : \mathrm{St}^{\mathrm{add}}(\mathcal{A})^{\mathrm{op}} \to \mathcal{S}p$, satisfying $\Qoppa|_{\mathcal{A}} \simeq iQ$. For each $x \in \mathcal{A}$ we have a fibre sequence

$$B_Q(x, x)_{hC_2} \to Q(x) \to \Lambda_Q(x),$$

so that connectivity of the restriction of $\mathcal{Q}$ to $\mathcal{A}$ is equivalent to connectivity of B_Q and Λ_Q.
Suppose now that $Q : \mathcal{E}^{op} \to \mathcal{S}p$ a quadratic functor on an exact ∞-category, such that Λ_Q and (for each $x \in \mathcal{E}$) the restricted polarisation $B_Q(-,x) : \mathcal{E}^{op} \to \mathcal{S}p$ lies in the connective aisle of $\mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})$, i.e. that the homotopy presheaves of Λ_Q and $B_Q(-,x)$ are effaceable in each negative degree. Note that the sheaf $B_Q(-,x)_{hC_2}$ is also connective with respect to this t-structure. From the fibre sequence

$$B_Q(x,x)_{hC_2} \to Q(x) \to \Lambda_Q(x)$$

for each $x \in \mathcal{E}$, we see that this is equivalent to effaceability of the negative homotopy presheaves of $Q(x)$ for each $x \in \mathcal{E}$. Given a quadratic left-exact functor $Q : \mathcal{E}^{op} \to \mathcal{S}p_{\geq 0}$, we may consider the composite

$$\mathrm{Fun}^{\mathrm{qlex}}(\mathcal{E}^{op}, \mathcal{S}p_{\geq 0}) \subset \mathrm{Fun}^{2-\mathrm{poly}}(\mathcal{E}^{op}, \mathcal{S}p) \simeq \mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}^{\mathrm{add}}(\mathcal{E})^{op}, \mathcal{S}p) \xrightarrow{\pi_!} \mathrm{Fun}^{2-\mathrm{exc}}(\mathrm{St}(\mathcal{E})^{op}, \mathcal{S}p) \overset{i^*}{\simeq} \mathrm{Fun}^{q}(\mathcal{E}^{op}, \mathcal{S}p),$$

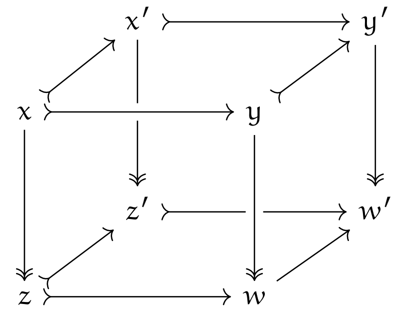
for $\pi : \mathrm{St}^{\mathrm{add}}(\mathcal{E}) \to \mathrm{St}(\mathcal{E})$ the Verdier quotient by the acyclics, and $i : \mathcal{E} \subset \mathrm{St}(\mathcal{E})$ the canonical inclusion. Write $Q : \mathcal{E}^{op} \to \mathcal{S}p$ for the pointwise inclusion into spectra, which is 2-polynomial as a functor on the underlying additive ∞-category $U\mathcal{E}$, and $\mathcal{Q} : \mathrm{St}^{\mathrm{add}}(\mathcal{E})^{op} \to \mathcal{S}p$ for the 2-excisive extension. We claim that canonical map $Q \to i^*\pi_!\mathcal{Q}$ truncates to an equivalence $Q(x) \simeq \tau_{\geq 0}(i^*\pi_!\mathcal{Q}(x))$ for each $x \in \mathcal{E}$.

Remark B.4.3. Suppose given an exact ∞-category $\mathcal{E}$ and a left-exact functor $F : \mathcal{E}^{op} \to \mathcal{S}p_{\geq 0}$. The pointwise extension $\iota F : \mathcal{E}^{op} \to \mathcal{S}p$ is an additive presheaf, with sheafification $(\iota F)^\dagger \in \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}^{op}, \mathcal{S}p)$ coinciding with the restriction to $\mathcal{E}$ of the 1-excisive approximation of $i_!\iota F : \mathrm{St}(\mathcal{E})^{op} \to \mathcal{S}p$, for $i : \mathcal{E} \subset \mathrm{St}(\mathcal{E})$ the canonical inclusion, and is computed as the filtered colimit

$$F^\dagger(x) \simeq \operatorname*{colim}_{y \twoheadrightarrow x} F(y).$$

The underlying $\mathcal{S}p_{\geq 0}$-valued functor of $(\iota F)^\dagger$ is pointwise $\tau_{\geq 0}\operatorname*{colim}_{y \twoheadrightarrow x} \iota F(y) \simeq \operatorname*{colim}_{y \twoheadrightarrow x} \tau_{\geq 0}\iota F(y) \simeq (\tau_{\geq 0}\iota F)^\dagger(x) \simeq F(x)$.

Remark B.4.4. Suppose $Q : \mathcal{E}^{op} \to \mathcal{S}p$ is a quadratic functor on an exact ∞-category, and consider a cube



in $\mathcal{E}$ in which each face is cocartesian (and hence bicartesian), inducing a map of fibre sequences of spectra

$$\begin{array}{ccccc} Q(w') & \longrightarrow & Q(z') \times_{Q(x')} Q(y') & \longrightarrow & B(\mathrm{fib}(x' \twoheadrightarrow z'), \mathrm{cofib}(x' \rightarrowtail y')) \\ \downarrow & & \downarrow & & \downarrow \\ Q(w) & \longrightarrow & Q(z) \times_{Q(x)} Q(y) & \longrightarrow & B(\mathrm{fib}(x \twoheadrightarrow z), \mathrm{cofib}(x \rightarrowtail y)), \end{array}$$

in which the rightmost arrow is an equivalence. We thus have that the square of spectra

$$\begin{array}{ccc} Q(w') & \longrightarrow & Q(z') \times_{Q(x')} Q(y') \\ \downarrow & & \downarrow \\ Q(w) & \longrightarrow & Q(z) \times_{Q(x)} Q(y) \end{array}$$

is bicartesian, i.e. the cube of spectra

$$
\begin{array}{ccccccc}
 & & Q(x') & & \longleftarrow & & Q(y') \\
 & \swarrow & \uparrow & & & \swarrow & \uparrow \\
Q(x) & & \longleftarrow & & Q(y) & & \\
\uparrow & & Q(z') & & \longleftarrow & & Q(w') \\
 & \swarrow & & & \uparrow & \nearrow & \\
Q(z) & & \longleftarrow & & Q(w) & &
\end{array}
$$

has vanishing total fibre, and is a limit diagram. Conversely, any functor $Q : \mathcal{E}^{op} \to \mathcal{S}p$ sending such cubes to limit diagrams is quadratic, by an application of [CDH$^+$I, Lem. 1.1.19] to the square

$$
\begin{array}{ccc}
x & \rightarrowtail & y \\
\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
0 & \rightarrowtail & z.
\end{array}
$$

Towards a hermitian theorem of the heart

C

In this section we include a sketch of a speculative approach to a theorem of the heart for the L-theory of stable ∞-categories equipped with bounded heart structures, closely following the approach of Harpaz in [HS25, App. A.1] for weight structures. After reviewing the formalism of heart structures, we consider the constructions of *loc. cit.* and prove some preliminary results.

C.1 Heart structures

We briefly recall the notion of a heart structure on a stable ∞-category from [Sau23].

Definition C.1.1 (Saunier)**.** A heart structure on a stable ∞-category $\mathcal{C}$ is the data of a pair $\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0} \subset \mathcal{C}$ of full subcategories satisfying the following conditions:

(i) $\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0}$ are closed under extensions, $\mathcal{C}_{\geq 0}$ under finite colimits, and $\mathcal{C}_{\leq 0}$ under finite limits.

(iii) For each $x \in \mathcal{C}$, there is a fibre sequence, the **heart decomposition**,

$$y \to x \to \Sigma z,$$

with $y \in \mathcal{C}_{\leq 0}$ and $z \in \mathcal{C}_{\geq 0}$.

We refer to the defining subcategories as the aisles of the heart structure, and may call objects in $\mathcal{C}_{\geq 0}$ resp. $\mathcal{C}_{\leq 0}$ connective resp. coconnective. For $a, b \in \mathbb{Z}$, write $\mathcal{C}_{\leq b} := \{x \in \mathcal{C} \mid \Omega^b x \in \mathcal{C}_{\leq 0}\}$, $\mathcal{C}_{\geq a} := \{x \in \mathcal{C} \mid \Omega^a x \in \mathcal{C}_{\geq 0}\}$, and $\mathcal{C}_{[a,b]} := \mathcal{C}_{\geq a} \cap \mathcal{C}_{\leq b}$ (the zero category for $a > b$). The associated heart is then $\mathcal{C}_{[0,0]} =: \mathcal{C}^{h\heartsuit}$. A stable ∞-category $\mathcal{C}$ with heart structure is **bounded** if for each $x \in \mathcal{C}$, there is some $n \geq 0$ with $x \in \mathcal{C}_{[-n,n]}$. A heart-exact functor $f : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories with heart structures is an exact functor on underlying stable ∞ categories preserving the respective subcategories.

Write $\mathrm{Cat}^{\mathrm{st}}_{h\heartsuit}$ for the ∞-category of small stable ∞-categories with heart structures and heart-exact functors between them, and $\mathrm{Cat}^{\mathrm{st}}_{h\heartsuit,\mathrm{bd}}$ for the full subcategory on bounded heart structures. Note that if we include the condition

(ii) for each $x \in \mathcal{C}_{\leq 0}$ and $y \in \mathcal{C}_{\geq 0}$, the mapping spectrum $\hom_{\mathcal{C}}(x, y)$ is connective,

and replace (i) with

(i)' $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$ are closed under retracts, $\mathcal{C}_{\geq 0}$ under finite colimits, and $\mathcal{C}_{\leq 0}$ under finite limits, we recover the usual definition of a weight structure, which (since it follows from axiom (ii) that the aisles of a weight structure are closed under extensions) is in particular a heart structure. The heart of a weight structure on a stable ∞-category is an additive ∞-category, in which moreover each fibre sequence splits by [HS25, Lem. 3.1.6], again owing to considerations stemming from axiom (ii). Discarding this axiom allows for non-split extensions in the heart: given a stable ∞-category with heart structure $\mathcal{C}$, we note that the heart $\mathcal{C}^{h\heartsuit} \subset \mathcal{C}$ is an extension-closed additive subcategory, which when equipped with the induced exact structure of Remark A.1.4 is the data of an exact ∞-category. The assignment $\mathcal{C} \mapsto \mathcal{C}^{h\heartsuit}$ sets up a functor

$$\mathcal{C}\mathrm{at}^{\mathrm{st}}_{h\heartsuit} \to \mathcal{E}\mathrm{xact}_\infty$$

which when restricted to $\mathcal{C}\mathrm{at}^{\mathrm{st}}_{h\heartsuit,\mathrm{bd}}$ is fully faithful, with essential image the full subcategory of weakly idempotent complete exact ∞-categories by [SW25, Th. 1.6]. The inverse to this functor is Klemenc's stable envelope, and there is an adjunction

$$\mathcal{E}\mathrm{xact}_\infty \underset{(-)^{h\heartsuit}}{\overset{\mathrm{St}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}\mathrm{at}^{\mathrm{st}}_{h\heartsuit}, \tag{C.1}$$

for which the unit $\mathcal{E} \to \mathrm{St}(\mathcal{E})^{h\heartsuit}$ is a weak idempotent completion. When restricted to the subcategory $\mathcal{C}\mathrm{at}^{\mathrm{st}}_{w\heartsuit} \subset \mathcal{C}\mathrm{at}^{\mathrm{st}}_{h\heartsuit}$ of weighted stable ∞-categories, this recovers the adjunction

$$\mathcal{C}\mathrm{at}^{\mathrm{add}}_\infty \underset{(-)^{h\heartsuit}}{\overset{\mathrm{St}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}\mathrm{at}^{\mathrm{st}}_{\mathrm{wt}}$$

of [Sos19, Cor. 3.4].

Lemma C.1.2. *Let $\mathcal{E}$ an exact ∞-category, equipped with the Grothendieck pretopology above. Then the sheafification $\mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \to \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})$ commutes with the truncation and connective cover functors.*

Proof. A sheaf $X \in \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})$ is by definition coconnective if its pointwise underlying infinite loopspace is discrete as a sheaf of spaces. Clearly, this property is preserved by the inclusion $\iota : \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \subset \mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})$, which is accordingly left t-exact. It follows formally that the left adjoint $\mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \to \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})$ is right t-exact, so that we have a commutative square

$$\begin{array}{ccc} \mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})_{\geq 0} & \xhookrightarrow{i} & \mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \\ \downarrow L & & \downarrow L \\ \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})_{\geq 0} & \xhookrightarrow{i'} & \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}). \end{array}$$

We claim that this diagram is right adjointable, i.e. that the composite natural transformation

$$L\tau_{\geq 0} \to \tau_{\geq 0} i' L\tau_{\geq 0} \simeq \tau_{\geq 0} L i \tau_{\geq 0} \to \tau_{\geq 0} L$$

is an equivalence. By full faithfulness of the restricted inlusion i', it suffices to show that for each spherical presheaf $X : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ and $x \in \mathcal{E}$, the canonical map

$$\tau_{\geq 0} L i \tau_{\geq 0} X(x) \to \tau_{\geq 0} L X(x)$$

is an equivalence, which follows since each of the truncations commutes with the filtered colimits computing the sheafification. In the diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})_{\leq 0} & \xhookrightarrow{j} & \mathcal{S}\mathrm{h}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \\ \downarrow \iota_{\leq 0} & & \downarrow \iota \\ \mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E})_{\leq 0} & \xhookrightarrow{j'} & \mathcal{P}^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{E}) \end{array}$$

witnessing left t-exactness of ι, the left vertical inclusion admits a left adjoint given by the composite $L_{\leq 0} := \tau_{\leq 0} \circ L \circ j'$, with counit

$$L_{\leq 0}\iota_{\leq 0} = \tau_{\leq 0} L j' \iota_{\leq 0} \simeq \tau_{\leq 0} L \iota j \to \tau_{\leq 0} j \to \mathrm{id}$$

induced by the counits $L\iota \to \mathrm{id}$ and $\tau_{\leq 0} j \to \mathrm{id}$, and unit given upon postcomposition with the fully faithful functor j' by

$$j' \to \iota L j' \to \iota j \tau_{\leq 0} L j' \simeq j' \iota_{\leq 0} \tau_{\leq 0} L j' = j' \iota_{\leq 0} L_{\leq 0},$$

induced by the units $\mathrm{id} \to \iota L$ and $\mathrm{id} \to j\tau_{\leq 0}$. Adjoining, we obtain a diagram

$$\begin{array}{ccc} \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})_{\leq 0} & \xleftarrow{\tau_{\leq 0}} & \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E}) \\ {\scriptstyle L_{\leq 0}}\uparrow & & \uparrow{\scriptstyle L} \\ \mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E})_{\leq 0} & \xleftarrow{\tau_{\leq 0}} & \mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E}). \end{array}$$

□

Warning C.1.3. The inclusion $\mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E}) \subset \mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E})$ is not right t-exact (unless $\mathcal{E}$ is split-exact).

Remark C.1.4. Given an exact ∞-category, recall that the stable envelope $\mathrm{St}(\mathcal{E})$ is defined as the Verdier quotient of the category $\mathrm{St}^{\mathrm{add}}(\mathcal{E}) = \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ by the subcategory of acyclics. The pair $(\mathrm{St}^{\mathrm{add}}_{\leq 0}(\mathcal{E}), \mathrm{St}^{\mathrm{add}}_{\geq 0}(\mathcal{E}))$ for $\mathrm{St}^{\mathrm{add}}_{\geq 0}(\mathcal{E})$ the essential image $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \hookrightarrow \mathrm{St}^{\mathrm{add}}(\mathcal{E})$ of the canonical functor into the Spanier-Whitehead stabilisation and $\mathrm{St}^{\mathrm{add}}_{\leq 0}(\mathcal{E})$ defined by orthogonality[1] is then a bounded weight structure on the additive stable envelope. Setting $\mathrm{St}_{\leq 0}(\mathcal{E})$ and $\mathrm{St}_{\geq 0}(\mathcal{E})$ to be the essential images of the respective weight aisles under the Verdier quotient by the acyclics, we obtain the (bounded) heart structure on $\mathrm{St}(\mathcal{E})$ of (C.1).

Remark C.1.5. The category of ind-objects $\mathrm{Ind}(\mathrm{St}(\mathcal{E}))$ identifies via the universal property of the stable envelope with the ∞-category $\mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})$ of spherical sheaves of spectra on $\mathcal{E}$ with respect to the Grothendieck topology generated by the covers $\{y \twoheadrightarrow x\}$ given by singleton families of egressions. The Verdier quotient $\mathrm{St}^{\mathrm{add}}(\mathcal{E}) \to \mathrm{St}(\mathcal{E})$ induces the sheafification $\mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E}) \to \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E})$, which is right t-exact by above. Accordingly, given a pointwise connective additive presheaf of spectra $\mathcal{F} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}p$, the sheafification $\mathcal{F}^\dagger$ has vanishing homotopy sheaves in negative degrees. Equivalently, the homotopy presheaves in negative degrees are weakly effaceable, in the sense that for $x \in \mathcal{E}$, $k \leq 0$, and $\alpha \in \pi_k\mathcal{F}(x)$, there is some egression $p : y \twoheadrightarrow x$ with $p^*\alpha = 0 \in \pi_k\mathcal{F}(y)$. For $x \in \mathcal{E} \subset \mathrm{St}(\mathcal{E})$, the restricted mapping spectrum $\hom_{\mathcal{E}}(-,x) = \hom_{\mathrm{St}(\mathcal{E})}(-,x)\mid_{\mathcal{E}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}p$ is obtained as the sheafification of the (pointwise connective) mapping spectrum $\hom_{\mathrm{St}^{\mathrm{add}}(\mathcal{E})}(-,x)\mid_{\mathcal{E}}$, and accordingly has weakly effaceable negative homotopy presheaves.

Lemma C.1.6. *Suppose given a diagram* $X : I \to \mathcal{P}^{\mathcal{S}p}(\mathcal{E})$, $i \mapsto X_i$, *such that for* $n < 0$, *for each* $i \in I$ *the presheaf* $\pi_n X_i$ *of abelian groups is weakly effaceable. Then the pointwise colimit* $\mathrm{colim}_I X_i$ *has effaceable negative homotopy presheaves.*

Proof. Lemma C.1.2 implies that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_\Sigma^{\mathcal{S}p}(\mathcal{E}) & \xrightarrow{L} & \mathcal{S}h_\Sigma^{\mathcal{S}p}(\mathcal{E}) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n^\dagger} \\ \mathcal{P}_\Sigma^{\mathcal{A}b}(\mathcal{E}) & \xrightarrow{L} & \mathcal{S}h_\Sigma^{\mathcal{A}b}(\mathcal{E}), \end{array}$$

[1] For any weighted stable ∞-category $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$, we have $\mathcal{C}_{\leq 0} = \{x \in \mathcal{C} \mid \hom_{\mathcal{C}}(x,y) \text{ is connective for all } y \in \mathcal{C}_{\geq 0}\}$, and dually $\mathcal{C}_{\geq 0} = \{x \in \mathcal{C} \mid \hom_{\mathcal{C}}(y,x) \text{ is connective for all } y \in \mathcal{C}_{\leq 0}\}$.

so that our assumption is equivalent to the statement that the composite $LX : I \to \mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$ takes values in connective sheaves of spectra. For $n < 0$, we have by closure of $\mathcal{S}h^{\mathcal{S}p}_{\Sigma}(\mathcal{E})$ under colimits that

$$\pi^{\dagger}_{n}(\operatorname{colim}_I X_i) \simeq \pi^{\dagger}_{n}(L \operatorname{colim}_I X_i) \simeq \pi^{\dagger}_{n}(\operatorname{colim}_I LX_i) \simeq 0.$$

□

C.2 Hermitian structures on exact ∞-categories

Let $\mathcal{E}$ an exact ∞-category, and recall from Appendix §B.4 that we call a functor $Q : \mathcal{E}^{op} \to \mathcal{S}p$ quadratic left exact if it satisfies the following:

(i) Q is reduced.

(ii) Q is additively quadratic, i.e. for each $x \in \mathcal{E}$, the functor

$$D_x Q : \mathcal{E}^{op} \to \mathcal{S}p, \quad y \mapsto \operatorname{fib}(Q(x \oplus y) \to Q(x) \oplus Q(y))$$

is exact (as a functor of exact categories).

(iii) for each congression $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{E}$, the map $Q(z) \to Q(y)$ exhibits $Q(z)$ as the total fibre of the square

$$\begin{array}{ccc} Q(y) & \longrightarrow & Q(x) \\ \downarrow & & \downarrow \\ B_Q(y,x) & \longrightarrow & B_Q(x,x) \end{array}$$

in $\mathcal{S}p$.

Write $\hom_{\mathcal{E}} : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{S}p$ for the restriction of the mapping spectra of $\operatorname{St}(\mathcal{E})$ along the inclusion $\mathcal{E}^{op} \times \mathcal{E} \subset \operatorname{St}(\mathcal{E})^{op} \times \operatorname{St}(\mathcal{E})$. We call a quadratic left exact functor $Q : \mathcal{E}^{op} \to \mathcal{S}p$ nondegenerate if there is a natural equivalence $B_Q(x,y) \simeq \hom_{\mathcal{E}}(x, \mathbb{D}_Q(y))$, for some exact equivalence $\mathbb{D}_Q : \mathcal{E}^{op} \to \mathcal{E}$ (as usual uniquely determined by Q). Write $\mathcal{E}xact^{\flat}_{\infty} \subset \mathcal{E}xact_{\infty}$ for the full subcategory of weakly idempotent-complete exact ∞-categories (the essential image of the functor $(-)^{h\heartsuit} : \mathcal{C}at^{st}_{\infty} \to \mathcal{E}xact_{\infty}$). Following [HS25, §3], we write $\mathcal{C}at^{eq}_{\flat}$ for subcategory of the cartesian unstraightening of the functor

$$(\mathcal{E}xact^{\flat}_{\infty})^{op} \to \mathcal{C}at_{\infty}, \quad \mathcal{E} \mapsto \operatorname{Fun}^{qlex}(\mathcal{E}^{op}, \mathcal{S}p)$$

spanned by the nondegenerate pairs $(\mathcal{E}, Q)$ with $Q : \mathcal{E}^{op} \to \mathcal{S}p$ quadratic left exact, with maps those pairs $(F : \mathcal{E} \to \mathcal{E}', \eta : Q \Rightarrow Q' \circ F^{op} : (\mathcal{E}, Q) \to (\mathcal{E}', Q')$, inducing via naturality a map (the image of $\mathrm{id}_{\mathbb{D}_Q(x)}$ under the map $\hom_{\mathcal{E}}(\mathbb{D}_Q(x), \mathbb{D}_Q(x)) \simeq B_Q(\mathbb{D}_Q(x), x) \to B_{Q'}(F(\mathbb{D}_Q(x)), F(x)) \simeq \hom_{\mathcal{E}'}(F(\mathbb{D}_Q(x)), \mathbb{D}_{Q'}(F(x))))$

$$\eta_{\#} : F(\mathbb{D}_Q(x)) \to \mathbb{D}_{Q'}(F(x))$$

which is an equivalence for each $x \in \mathcal{E}$. Call a pair $(\mathcal{E}, Q) \in \mathcal{C}at^{eq}_{\flat}$ an exact Poincaré category. Under the equivalence of Lemma B.4.1, Q extends uniquely to a 2-excisive functor $\mathcal{Q} : \operatorname{St}(\mathcal{E})^{op} \to \mathcal{S}p$ which is moreover reduced, with $B_{\mathcal{Q}}|_{\mathcal{E}^{op} \times \mathcal{E}^{op}} \simeq B_Q$.

Remark C.2.1. By currying, we have an equivalence

$$\mathrm{Fun}^{\mathrm{biex}}(\mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}^{\mathrm{biex}}(\mathrm{St}(\mathcal{E})^{\mathrm{op}} \times \mathrm{St}(\mathcal{E})^{\mathrm{op}}, \mathcal{S}\mathrm{p}),$$

and we note that $\mathbb{D}_Q : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ induces a unique extension $\mathbb{D} : \mathrm{St}(\mathcal{E})^{\mathrm{op}} \to \mathrm{St}(\mathcal{E})$. The functors $\hom_{\mathrm{St}(\mathcal{E})}(-, \mathbb{D}(-))$ and $B_{\mathfrak{P}}(-,-) : \mathrm{St}(\mathcal{E})^{\mathrm{op}} \times \mathrm{St}(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ coincide on $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}}$, and accordingly we have a natural equivalence $B_{\mathfrak{P}}(-,-) \simeq \hom_{\mathrm{St}(\mathcal{E})}(-, \mathbb{D}(-))$, and pair the $(\mathrm{St}(\mathcal{E}), \mathfrak{P})$ is a Poincaré category. Under the equivalence $\mathcal{C}\mathrm{at}^{\mathrm{st}}_{h\heartsuit,\mathrm{bd}} \to \mathcal{E}\mathrm{xact}^{\flat}_{\infty}$ of [SW25, Th. 1.6], since $\mathbb{D}_{\mathfrak{P}}$ preserves the heart by construction, it is heart exact, and accordingly restricts to equivalences $\mathrm{St}(\mathcal{E})^{\mathrm{op}}_{[a,b]} \simeq \mathrm{St}(\mathcal{E})_{[-b,-a]}$ for each $a, b \in \mathbb{Z}$. As in [HS25, Th. 3.2.5], the functor $(-)^{h\heartsuit}$ induces an equivalence of ∞-categories

$$\mathcal{C}\mathrm{at}^{\mathrm{p}}_{h\heartsuit} \to \mathcal{E}\mathrm{xact}^{\mathrm{eq}}_{\flat}, (\mathcal{C}, \mathfrak{P}) \mapsto (\mathcal{C}^{h\heartsuit}, \mathfrak{P}|_{\mathcal{C}^{h\heartsuit}}),$$

for $\mathcal{C}\mathrm{at}^{\mathrm{p}}_{h\heartsuit}$ the ∞-category of Poincaré ∞-categories equipped with bounded heart structures such that the duality $\mathbb{D}_{\mathfrak{P}}$ preserves the heart, and Poincaré functors whose underlying functors are heart-exact.

For $(\mathcal{E}, Q) \in \mathcal{C}\mathrm{at}^{\mathrm{eq}}_{\flat}$, write $Q_{\mathrm{nd}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ for the functor given pointwise by the pullback

$$\begin{array}{ccc} Q_{\mathrm{nd}}(x) & \longrightarrow & \Omega^{\infty} Q(x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{E}^{\simeq}}(x, \mathbb{D}_Q(x)) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(x, \mathbb{D}_Q(x)), \end{array}$$

where the map $\Omega^{\infty}(x) \to \mathrm{Map}_{\mathcal{E}}(x, \mathbb{D}_Q(x))$ is induced as usual by the composite

$$Q(x) \xrightarrow{\nabla} Q(x \oplus x) \to B_Q(x,x) \simeq \hom_{\mathcal{E}}(x, \mathbb{D}_Q(x)).$$

Set $\mathrm{Pn}(\mathcal{E}, Q) := \int_{(\mathcal{E}^{\simeq})^{\mathrm{op}}} Q_{\mathrm{nd}}$, the moduli space of nondegenerate forms in $(\mathcal{E}, Q)$.

C.3 L-theory for exact ∞-categories

Given an exact Poincaré category $(\mathcal{E}, Q)$, write $\mathcal{Q}_n(\mathcal{E}) \subset \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{E})$ for the full subcategory spanned by functors $X : \mathrm{TwAr}(\Delta^n) \to \mathcal{E}$ such that for each $i \leq j \leq k \leq l$, the square

$$\begin{array}{ccc} X_{il} & \twoheadrightarrow & X_{jl} \\ \downarrow & & \downarrow \\ X_{ik} & \twoheadrightarrow & X_{jk} \end{array}$$

is ambigressive, and define $Q_n : \mathcal{Q}_n(\mathcal{E})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ via

$$Q_n(X) := \lim_{\mathrm{TwAr}(\Delta^n)^{\mathrm{op}}} Q \circ X^{\mathrm{op}}.$$

The ∞-category $\mathcal{Q}_n(\mathcal{E})$ carries an exact structure in which the ingressions and egressions are defined pointwise, with respect to which the functor Q_n is quadratic left exact and nondegenerate (since this is the case in the stable setting [CDH+II, Lem. 2.2.6], and the functor $i_* : \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathcal{E}) \to \mathrm{Fun}(\mathrm{TwAr}(\Delta^n), \mathrm{St}(\mathcal{E}))$ restricts to an exact embedding $\mathcal{Q}_n(\mathcal{E}) \to \mathcal{Q}_n(\mathrm{St}(\mathcal{E}))$, with $\mathfrak{P}_n|_{\mathcal{Q}_n(\mathcal{E})} \simeq Q_n$, and such that the duality induced by $\mathfrak{P}_n$ preserves $\mathcal{Q}_n(\mathcal{E})$). The exact Poincaré categories $\mathcal{Q}_n(\mathcal{E}, Q)$ thus obtained compile into a simplicial exact Poincaré category $\mathcal{Q}_\bullet(\mathcal{E}, Q)$, and we define $\mathcal{Q}^{(n)}(\mathcal{E}, Q)$ to be the multisimplicial exact Poincaré category given in multidegree $(k_1, \dots, k_n)$ by $\mathcal{Q}_{k_1} \dots \mathcal{Q}_{k_n}(\mathcal{E}, Q)$. Setting $\mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{E}, Q)) := \operatorname{colim}_{(k_1,\dots,k_n)\in(\Delta^{\times n})^{op}} \mathrm{Pn}\mathcal{Q}_{k_1} \dots \mathcal{Q}_{k_n}(\mathcal{E}, Q)$, we define the L-theory space, following [Sch10b, Rem. 8] and [HS25, App. A.1], as

$$\mathcal{L}(\mathcal{E}, Q) := \operatorname{colim}_n \mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{E}, Q)),$$

with the maps $\mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{E}, Q)) \to \mathrm{Pn}(\mathcal{Q}^{(n+1)}(\mathcal{E}, Q))$ the inclusion of vertices, i.e. induced by the identification $\mathcal{Q}_0 = \mathrm{id}$ in the exterior $\mathcal{Q}$-term.

Remark C.3.1. The presence of an exact structure introduces a choice in the definition of the $\mathcal{Q}$-construction independent of that of Remark 2.3.2: given a triple $0 \le i \le j \le k \le n$ and $X \in \mathcal{Q}_n(\mathcal{C})$, we may require the map $X_{il} \to X_{jl}$ to be an egression and $X_{il} \to X_{ik}$ an ingression, or conversely. Denoting these respectively by $\mathcal{Q}^l_n(\mathcal{C})$ resp. $\mathcal{Q}^r_n(\mathcal{C})$, we see that precomposition with the map $\mathrm{TwAr}(\Delta^n) \simeq \mathrm{TwAr}(\Delta^n)$ induced by $[n]^{op} \to [n]$, $i \mapsto n - i$ induces an equivalence $\mathcal{Q}^l_n(\mathcal{C}) \simeq \mathcal{Q}^r_n(\mathcal{C})$, sending $X \mapsto (\widetilde{X} : (i \le j) \mapsto X_{n-j,n-i})$, sending an arrow

$$\begin{array}{ccc} i & \longrightarrow & l \\ \downarrow & & \uparrow \\ j & \longrightarrow & k \end{array}$$

to either composite

$$\begin{array}{ccc} \widetilde{X}_{il} = X_{n-l,n-i} & \rightarrowtail & X_{n-l,n-j} = \widetilde{X}_{jl} \\ \downarrow & & \downarrow \\ \widetilde{X}_{ik} = X_{n-k,n-i} & \rightarrowtail & X_{n-k,n-j} = \widetilde{X}_{jk} \end{array}$$

Remark C.3.2. For $(\mathcal{C}, \Qoppa) \in \mathrm{Cat}^p_\infty$, the exact sequence of commutative monoids

$$\pi_0 \mathrm{Pn}\,\mathrm{Met}(\mathcal{C}, \Qoppa) \to \pi_0 \mathrm{Pn}(\mathcal{C}, \Qoppa) \to \mathrm{coeq}(\pi_0 \mathrm{Pn}(\mathcal{Q}_1(\mathcal{C}, \Qoppa)) \underset{d_0}{\overset{d_1}{\rightrightarrows}} \pi_0 \mathrm{Pn}(\mathcal{C}, \Qoppa)) \to 0$$

of [CDH+II, Prop. 2.3.7] implies that $\mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{E}, Q))$ is grouplike, furnishing equivalences

$$\mathcal{L}(\mathcal{C}, \Qoppa) \simeq \left(\operatorname{colim}_n \mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa))\right)^{grp} \simeq \operatorname{colim}_n \mathcal{GWQ}^{(n)}(\mathcal{C}, \Qoppa) \simeq \operatorname{colim}_n \Omega^{\infty-n}\mathrm{GW}(\mathcal{C}, \Qoppa^{[-n]}) \simeq \Omega^\infty \mathrm{L}(\mathcal{C}, \Qoppa)$$

by the constructions of [CDH+II].

For $(\mathcal{C}, \Qoppa)$ a Poincaré category with bounded heart structure for which $\mathbb{D}_\Qoppa$ restricts to $\mathcal{C}^{h\heartsuit}$, the exact Poincaré functor $(\mathcal{C}^{h\heartsuit}, \Qoppa^{h\heartsuit} := \Qoppa|_{\mathcal{C}^{h\heartsuit}}) \to (\mathcal{C}, \Qoppa)$ induces a map of spaces

$$\mathcal{L}(\mathcal{C}^{h\heartsuit}, \Qoppa^{h\heartsuit}) \to \mathcal{L}(\mathcal{C}, \Qoppa), \tag{C.2}$$

which when $\mathcal{C}$ is weighted (equivalently, the exact structure on the heart $\mathcal{E}$ is split) is an equivalence when Q takes values in connective spectra, by [HS25, Th. A.1.2].

C.4 Algebraic surgery

In this section we give a brief review of algebraic surgery for Poincaré categories. Recall firstly that associated to $(\mathcal{C}, \Qoppa) \in \mathrm{Cat}^p_\infty$ we have the cobordism category $\mathrm{Cob}(\mathcal{C}, \Qoppa) := \mathrm{acat}\left(\mathrm{Pn}(\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa^{[1]}))\right)$ (note the shifted Poincaré structure), with objects those of $\mathrm{Pn}(\mathcal{C}, \Omega^{[1]})$, and maps informally given by spans $(x,q) \xleftarrow{f} (z,\eta) \xrightarrow{g} (x',q')$, with $\eta : f^*q \simeq g^*q' \in \Omega^\infty \Omega^{[1]}(z)$, such that the square

$$\begin{array}{ccc} w & \longrightarrow & x' \simeq \Sigma\,\mathbb{D}_\Omega(x') \\ \downarrow & & \downarrow \\ x \simeq \Sigma\,\mathbb{D}_\Omega(x) & \longrightarrow & \Sigma\,\mathbb{D}_\Omega(w) \end{array}$$

furnished by η is cartesian. In this case, we call (x,q) and (x',q') cobordant. We view such a span as a map from (x,q) to (x',q'), which we write as $(x,q) \rightsquigarrow (x',q')$.

To any ∞-category K we may associate the K-indexed hermitian $\mathcal{Q}$-construction $\mathcal{Q}_K(\mathcal{C}, \Omega)$ (see §2.5). Taking Poincaré objects, this compiles into a functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{S}$, $K \mapsto \mathrm{Pn}(\mathcal{Q}_K(\mathcal{C},\Omega))$. Now given a simplicial space $X : \Delta^{\mathrm{op}} \to \mathcal{S}$, it follows from the pointwise formula that the right Kan extension of X along the inclusion $i : \Delta^{\mathrm{op}} \subset \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$ is given by

$$i_*X(K) = \lim_{\Delta^n \to K} X_n \simeq \lim_{N(\Delta^n)\to N(K)} \mathrm{Map}_{s\mathcal{S}}(N(\Delta^n), X) \simeq \mathrm{Map}_{s\mathcal{S}}(\operatorname*{colim}_{N(\Delta^n)\to N(K)} N(\Delta_n), X) \simeq \mathrm{Map}_{s\mathcal{S}}(N(K), X),$$

where $N : \mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$ is the Rezk nerve. If X is restricted from a functor $\widetilde{X} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{S}$ there is by definition a natural transformation $\widetilde{X} \Rightarrow i_*X$. Applying this to $\mathrm{Pn}(\mathcal{Q}_{(-)}(\mathcal{C}, \Omega^{[1]}))$, there is for each $K \in \mathcal{C}\mathrm{at}_\infty$ a map

$$\mathrm{Pn}(\mathcal{Q}_K(\mathcal{C}, \Omega^{[1]})) \to \mathrm{Hom}_{s\mathcal{S}}(N(K), \mathrm{Pn}(\mathcal{Q}_\bullet(\mathcal{C}, \Omega^{[1]}))),$$

which postcomposing with the completion functor and adjoining via acat $\dashv$ N gives rise to a map

$$\mathrm{Pn}(\mathcal{Q}_K(\mathcal{C}, \Omega^{[1]})) \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\infty}(K, \mathrm{Cob}(\mathcal{C}, \Omega)), \tag{C.3}$$

which is an equivalence by [CDH+II, Prop. 2.3.6]. This implies the description above of the arrows of $\mathrm{Cob}(\mathcal{C}, \Omega)$. In particular, a map $0 \rightsquigarrow (x,q)$ in $\mathrm{Cob}(\mathcal{C}, \Omega)$ is a map $f : L \to x$ and a nullhomotopy of $f^*q \in \Omega^\infty \Omega^{[1]}(L)$, such that the induced nullhomotopy of the composite

$$L \to x \simeq \Sigma\,\mathbb{D}_\Omega(x) \to \Sigma\,\mathbb{D}_\Omega(L)$$

renders this an exact sequence in $\mathcal{C}$. If we weaken the latter requirement we arrive at the definition of a surgery datum for (x,q) (relative to $\Omega^{[1]}$), i.e. a map $f : L \to x$ and a nullhomotopy $\eta : f^*q \simeq 0 \in \Omega^\infty \Omega^{[1]}(L)$.

Construction C.4.1. Suppose given a surgery datum $(f, \eta : L \to (x,q))$ with respect to $(\mathcal{C}, \Omega)$, and consider the following diagram of exact sequences:

$$\begin{array}{ccccc} L & = & L & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow \\ \chi(f) & \longrightarrow & X & \longrightarrow & \mathbb{D}_\Omega(L) \\ \downarrow & & \downarrow & & \| \\ X_f & \longrightarrow & X/L & \longrightarrow & \mathbb{D}_\Omega(L), \end{array}$$

where commutativity of the upper right-hand square is witnessed by the nullhomotopy η. The object X_f is the **result of the surgery**, and it follows from an argument using quadraticity of Ω that the restriction of $q \in \Omega^\infty \Omega(X)$ to $\Omega^\infty \Omega(\chi(f))$ lifts to a Poincaré form $q' \in \Omega^\infty \Omega(X_f)$, i.e. we obtain a map $X \rightsquigarrow X_f$ in $\mathrm{Cob}(\mathcal{C}, \Omega^{[-1]})$.

A pair $(f : L \to x, \eta : f^*q \simeq 0)$ is precisely a hermitian object in $\mathrm{Met}(\mathcal{C}, \Omega^{[1]})$ whose image under the target map $\mathrm{He}(\mathrm{Met}(\mathcal{C}, \Omega^{[1]})) \to \mathrm{He}(\mathcal{C}, \Omega^{[1]})$ is the Poincaré object (x, q), and we write $\mathrm{Surg}(\mathcal{C}, \Omega)$ for the pullback

$$\begin{array}{ccc} \mathrm{Surg}(\mathcal{C}, \Omega) & \hookrightarrow & \mathrm{He}(\mathrm{Met}(\mathcal{C}, \Omega)) \\ \downarrow & & \downarrow \\ \mathrm{Pn}(\mathcal{C}, \Omega) & \hookrightarrow & \mathrm{He}(\mathcal{C}, \Omega). \end{array}$$

For $(x, q) \in \mathrm{Pn}(\mathcal{C}, \Omega)$, write $\mathrm{Surg}(\mathcal{C}, \Omega)_{(x,q)}$ for the fibre of the left-hand map over (x, q), equivalently given as the total space of the right fibration classified by $\mathcal{C}_{/x} \to \mathcal{S}$, $(f : z \to x) \mapsto \{0\} \times_{\Omega^\infty \Omega(z)} \{f^*q\}$. Given $(x, q) \in \mathrm{Pn}(\mathcal{C}, \Omega)$, there is a **surgery equivalence**

$$\mathrm{Cob}(\mathcal{C}, \Omega^{[-1]})_{(x,q)/} \to \mathrm{Surg}(\mathcal{C}, \Omega)_{(x,q)} \tag{C.4}$$

witnessing that the target of a Poincaré span is the result of surgery along the inclusion of the fibre of the right leg into x. Informally, a span $(x, q) \xleftarrow{i} (w, \eta) \xrightarrow{p} (x', q')$ is sent to to $(f : \mathrm{fib}(p) \to x, \eta')$, with η' induced by $\eta : i^*q \simeq p^*q'$; the inverse takes a surgery datum $(f : L \to x, \eta : f^*q \simeq 0)$ to the span $x_f \leftarrow \chi(f) \to x$ of construction C.4.1.

Since by [CDH⁺III, Prop. 1.1.13] cobordant objects have the same class in L-theory, to control L-groups we may, starting with some class represented by some (x, q) with $x \in \mathcal{C}_{[-m,m]}$, hope via iterative surgery along judiciously chosen maps to exhibit (x, q) as cobordant to increasingly bounded objects, thus relating the L-groups of $(\mathcal{C}, \Omega)$ to those of its heart.

C.5 Heart surgery?

We start with the following straightforward generalisation of [HS25, Lem. A.1.3].

Lemma C.5.1. *For* $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}_{h\heartsuit}$ *and* $n \geq 0$*, the* ∞*-categories* $\mathcal{Q}_n(\mathcal{C})$ *acquire heart structures in which* $X \in \mathcal{Q}_n(\mathcal{C})_{[a,b]}$ *if and only if it satisfies the following:*

(i) $X_{ij} \in \mathcal{C}_{[a,b]}$ *for each* $0 \leq i \leq j \leq n$*;*

(ii) for $0 \leq i \leq j \leq k \leq n$, $\mathrm{fib}(X_{ik} \twoheadrightarrow X_{jk})$ *and* $\mathrm{cofib}(X_{ik} \to X_{ij})$ *lie in* $\mathcal{C}_{[a,b]}$.

This heart structure is moreover bounded if that on $\mathcal{C}$ *is.*

Proof. Modulo closure under extensions, this follows from the analogous statement for weight structures [HS25, Lem. A.1.3]. We reproduce the proof of existence of heart decompositions to make clear that these are independent of the connectivity axiom (ii): under the equivalences $\mathcal{Q}_n(\mathcal{C}) \simeq S_{2n+1}(\mathcal{C}) \simeq \mathrm{Fun}(\Delta^{2n}, \mathcal{C})$, it suffices to show inductively that $\mathrm{Fun}(\Delta^m, \mathcal{C})$ admits a heart structure with connective objects the pointwise connective sequences $x_0 \to \cdots \to x_m$, and coconnective objects pointwise coconnective sequences with $\mathrm{cofib}(x_i \to x_{i+1}) \in \mathcal{C}_{\leq 0}$ for each i. For $m = 0$, we may use the heart decomposition of $\mathcal{C}$, so suppose $m \geq 1$ and such decompositions exist for $n \leq m-1$. Given $(x_0 \to \cdots \to x_{m-1} \xrightarrow{f_{m-1}} x_m) \in \mathrm{Fun}(\Delta^m, \mathcal{C})$, we choose a decomposition

$$\begin{array}{ccccccc} y_0 & \longrightarrow & \cdots & \longrightarrow & y_{m-1} & & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i_{m-1}} & & \\ x_0 & \longrightarrow & \cdots & \longrightarrow & x_{m-1} & \longrightarrow & x_m \\ \downarrow & & \downarrow & & \downarrow & & \\ z_0 & \longrightarrow & \cdots & \longrightarrow & z_{m-1} & & \end{array} \tag{C.5}$$

with $(y_i)_i \in \mathrm{Fun}(\Delta^{m-1}, \mathcal{C})_{\leq 0}$ and $(z_j)_j \in \mathrm{Fun}(\Delta^{m-1}, \mathcal{C})_{\geq 0}$, and set $x' := \mathrm{cofib}(f_{m-1} i_{m-1})$. Choosing a heart decomposition $y' \to x' \to z_m$ with $y' \in \mathcal{C}_{\leq 0}$ and $z_m \in \mathcal{C}_{\geq 1}$, we note that the canonical nullhomotopy of the composite map $y_{m-1} \to z_m$ furnishes a lift $y_{m-1} \to \mathrm{fib}(x_m \to z_m) =: y_m$, participating in the exact sequence

$$y_{m-1} \simeq \mathrm{fib}(x_m \to x') \to y_m \to \mathrm{fib}(x' \to z_m) \simeq y'$$

by Corollary A.2.2. This exhibits y_m as an extension of coconnectives and hence coconnective, with moreover the cofibre of the map $y_{m-1} \to y_m$ coconnective.

Given a congression $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{Q}_n(\mathcal{C})$ with $X, Z \in \mathcal{C}_{[a,b]}$, then $Y_{ij} \in \mathcal{C}_{[a,b]}$ by closure under extensions of $\mathcal{C}_{[a,b]}$ applied to $X_{ij} \rightarrowtail Y_{ij} \twoheadrightarrow Z_{ij}$. For $0 \leq i \leq j \leq k \leq n$, we have a map of congressions

$$\begin{array}{ccccc} X_{ik} & \rightarrowtail & Y_{ik} & \twoheadrightarrow & Z_{ik} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X_{jk} & \rightarrowtail & Y_{jk} & \twoheadrightarrow & Z_{jk}, \end{array}$$

inducing by Lemma A.2.4 a congression $\mathrm{fib}(f) \rightarrowtail \mathrm{fib}(g) \twoheadrightarrow \mathrm{fib}(h)$, and so $\mathrm{fib}(g) \in \mathcal{C}_{[a,b]}$ by closure under extensions. The dual argument works for cofibres. □

We now sketch the surgery argument of [HS25, App. A.1] demonstrating that (C.2) is an equivalence in the case of a Poincaré category with bounded weight structure such that $\Qoppa|_{\mathcal{C}^{\heartsuit}}$ takes connective values, and the duality preserves the heart, and speculate as to a possible extension to bounded heart structures. All definitions and constructions below are from *loc. cit.*. Fix a Poincaré category with heart structure $(\mathcal{C}, \Qoppa, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that $\mathbb{D}_{\Qoppa}$ preserves the heart.

Definition C.5.2. For $m \geq 0$, call an object $x \in \mathcal{C}$ m-bounded if $x \in \mathcal{C}_{[-m,m]}$, and a Poincaré object $(x, q \in \Omega^\infty \Qoppa(x))$ m-bounded if the underlying object is. Write $\mathrm{Pn}_m(\mathcal{C}, \Qoppa) \subset \mathrm{Pn}(\mathcal{C}, \Qoppa)$ for the full subspace on the m-bounded Poincaré objects, and

$$\mathrm{Pn}_m(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa)) = \operatorname*{colim}_{(k_1,\dots,k_n) \in (\Delta^{\times n})^{\mathrm{op}}} \mathrm{Pn}_m(\mathcal{Q}_{k_1} \dots \mathcal{Q}_{k_n}(\mathcal{C}, \Qoppa)),$$

where we iteratively use the induced heart structures above on $\mathcal{Q}_n(\mathcal{C}, \Qoppa)$.

Remark C.5.3. (i) Under the equivalence $\mathcal{Q}_n(\mathcal{C}) \simeq S_{2n+1}(\mathcal{C})$, an object $X \in S_{2n+1}(\mathcal{C})_{[a,b]}$ precisely when each $X_{ij} \in \mathcal{C}_{[a,b]}$. If $(\mathcal{C}, \Qoppa)$ is Poincaré, an object $(x, q) \in \mathrm{Pn}(\mathcal{C}, \Qoppa)$ is m-bounded if and only if x is $(-m)$-connective, if and only if it is m-coconnective. Accordingly, a span $(x_{00}, q_{00}) \xleftarrow{f_{00}} (x_{01}, \eta) \xrightarrow{f_{11}} (x_{11}, q_{11})$ lies in $\mathrm{Pn}_m(\mathcal{Q}_1(\mathcal{C}, \Qoppa))$ if and only if x_{00} and $x_{11} \in \mathcal{C}_{\geq -m}$ and the map $x_{01} \to x_{11}$ is $(-m)$-connected (has fibre in $\mathcal{C}_{\geq -m}$), if and only if x_{00} and x_{11} are m-coconnective and $x_{01} \to x_{00}$ is m-coconnected (has cofibre in $\mathcal{C}_{\leq m}$).

(ii) Given an m-bounded Poincaré object $(x, q) \xleftarrow{i} (w, \eta) \xrightarrow{p} (x', q')$ of $\mathcal{Q}_1(\mathcal{C}, \Qoppa)$ corresponding to a map $(x, q) \rightsquigarrow (x', q')$ by (C.3) and hence an object of $\mathrm{Cob}(\mathcal{C}, \Qoppa)_{(x,q)}$, then the corresponding surgery datum $(f : L \to x, \eta : f^*q \simeq 0)$ arising from (C.4) has L m-bounded.

The idea behind the proof of [HS25, Th. A.1.2] is to show that the square

$$\begin{array}{ccc} \mathrm{Pn}_m(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}^{(n+1)}(\mathcal{C}, \Qoppa)) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Pn}_{m+1}(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{Q}^{(n+1)}(\mathcal{C}, \Qoppa)) \end{array} \tag{C.6}$$

admits dashed lifts as shown for each $m, n \geq 0$. In this case, it follows that the maps $\mathrm{colim}_n \mathrm{Pn}_m(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa)) \to \mathrm{colim}_n \mathrm{Pn}_{m+1}(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa))$ are equivalences[2], so that the transfinite composition

$$\mathcal{L}(\mathcal{C}^{h\heartsuit}, \Qoppa^{h\heartsuit}) = \mathrm{colim}_n \mathrm{Pn}_0 \mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa) \to \mathrm{colim}_n \mathrm{Pn}_1 \mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa) \to \cdots \to \mathrm{colim}_n \mathrm{Pn}(\mathcal{Q}^{(n)}(\mathcal{C}, \Qoppa)) = \mathcal{L}(\mathcal{C}, \Qoppa)$$

is an equivalence. To this end, write $\mathfrak{P}$ $(\mathfrak{P}^\circ) : \Delta \to \mathrm{Poset}$ for the functor sending $[n]$ to the set of (nonempty) subsets of $[n]$ under inclusion. There is a span of natural transformations with components at $[n]$ given by $[n] \Leftarrow \mathfrak{P}^\circ \Rightarrow \mathfrak{P}$, for the left leg $S \mapsto \max(S)$, and the right leg the inclusion. This induces a square of simplicial objects in $\mathrm{Cat}_\infty^{\mathrm{p}}$

$$\begin{array}{ccc} (\mathcal{C}, \Qoppa) & \longrightarrow & \mathcal{Q}_\bullet(\mathcal{C}, \Qoppa) \\ \downarrow & & \downarrow \\ \mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C}, \Qoppa) & \longrightarrow & \mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C}, \Qoppa), \end{array}$$

for the upper-left corner the constant diagram, and accordingly a square of simplicial spaces

$$\begin{array}{ccc} \mathrm{Pn}(\mathcal{C}, \Qoppa) & \longrightarrow & \mathrm{Pn}(\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa)) \\ \downarrow & & \downarrow \\ \mathrm{Pn}(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C}, \Qoppa)). \end{array}$$

Call an object $X \in \mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C})$ $(\in \mathcal{Q}_{\mathfrak{P}^\circ([n])}(\mathcal{C}))$ locally m-bounded if for each simplex $[k] \to \mathfrak{P}([n])$ $([k] \to \mathfrak{P}^\circ([n]))$, the image of X in $\mathcal{Q}_k(\mathcal{C})$ is m-bounded with respect to the above heart structure. Write $\mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C}, \Qoppa))$ and $\mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}^\circ([n])}(\mathcal{C}, \Qoppa))$ for the full subspaces on the Poincaré objects whose underlying objects are locally m-bounded, so that there is a square

$$\begin{array}{ccc} \mathrm{Pn}_m(\mathcal{C}, \Qoppa) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa)) \\ \downarrow & & \downarrow \\ \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C}, \Qoppa)) \end{array} \tag{C.7}$$

for each $m \geq 0$. One then observes that we have by naturality a commutative square of simplicial spaces

$$\begin{array}{ccccc} & & \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C}, \Qoppa)), \\ & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathrm{Pn}_m(\mathcal{C}, \Qoppa) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa)) & & \\ \downarrow & & \mathrm{Pn}_{m+1}(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{C}, \Qoppa) \\ & \nearrow & \downarrow & \nearrow & \\ \mathrm{Pn}_{m+1}(\mathrm{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C}, \Qoppa)) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{Q}_\bullet(\mathcal{C}, \Qoppa)) & & \end{array}$$

[2] Indeed, the sequential diagrams $\mathrm{Pn}_m(\mathcal{C}, \Qoppa) \to \mathrm{Pn}_m(\mathcal{Q}^{(1)}(\mathcal{C}, \Qoppa)) \to \ldots$ and $\mathrm{Pn}_{m+1}(\mathcal{C}, \Qoppa) \to \mathrm{Pn}_{m+1}(\mathcal{Q}^{(1)}(\mathcal{C}, \Qoppa)) \to \ldots$ are each cofinal subdiagrams of $\mathrm{Pn}_m(\mathcal{C}, \Qoppa) \to \mathrm{Pn}_{m+1}(\mathcal{C}, \Qoppa) \to \mathrm{Pn}_m(\mathcal{Q}^{(1)}(\mathcal{C}, \Qoppa)) \to \mathrm{Pn}_{m+1}(\mathcal{Q}^{(1)}(\mathcal{C}, \Qoppa)) \to \mathrm{Pn}_m(\mathcal{Q}^{(2)}(\mathcal{C}, \Qoppa)) \ldots$.

such that the inward-pointing maps induce equivalences upon realisation, the two rightmost since $\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C},\mathfrak{Q})$ is the simplicial subdivision (the image under $\mathrm{Ex}^\infty : s\mathcal{S} \to s\mathcal{S}$) of $\mathcal{Q}_\bullet(\mathcal{C},\mathfrak{Q})$, and the leftmost by [HS25, Lem. A.1.12]. To exhibit a lift in (C.6), it suffices to do so for the square of simplicial spaces

$$\begin{array}{ccc} \mathrm{Pn}_m(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}^{(1)}_\bullet(\mathcal{C},\mathfrak{Q})) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Pn}_{m+1}(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{Q}^{(1)}_\bullet(\mathcal{C},\mathfrak{Q})), \end{array}$$

or equivalently for the square (C.7). We sketch the argument of [HS25, Lem. A.1.7, A.1.13]:

(i) By the equivalence (C.3), a Poincaré object (X,ξ) of $\mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C},\mathfrak{Q})$ corresponds to a diagram $\phi_{(X,q)} : \mathfrak{P}([n]) \to \mathrm{Cob}(\mathcal{C},\mathfrak{Q}^{[-1]})$, and moreover if X is locally m-bounded, this will take values in the subcategory $\mathrm{Cob}^{m,m}(\mathcal{C},\mathfrak{Q}^{[-1]})$ of m-bounded Poincaré objects with m-bounded maps between them, i.e. those spans $(x,q) \xleftarrow{i} (w,\eta) \xrightarrow{p} (x',q')$ with $x, x', \mathrm{fib}(p), \mathrm{cofib}(i) \in \mathcal{C}_{[-m,m]}$.

(ii) Viewing $\xi \in \lim_{\mathrm{TwAr}(\mathfrak{P}([n]))^{op}} \Omega^\infty \mathfrak{Q} \circ X^{op}$ as a coherently compatible family of points $\xi_{U\subset V} \in \Omega^\infty\mathfrak{Q}(X_{U\subset V})$ and writing $(x,q) := (X_{\varnothing\subset\varnothing}, \xi_{\varnothing\subset\varnothing})$, we obtain a diagram $\mathfrak{P}([n]) \to \mathrm{Cob}(\mathcal{C},\mathfrak{Q}^{-1]})_{(x,q)/}$, which under the equivalence (C.3) corresponds to a diagram

$$\psi_{(X,\xi)} : \mathfrak{P}([n]) \to \mathrm{Surg}(\mathcal{C},\mathfrak{Q})_{(x,q)}$$

of surgery data for (x,q), sending $\varnothing$ to the trivial (zero) surgery datum.

(iii) Define the **surgery complex** $\mathrm{SC}^m_n(\mathcal{C},\mathfrak{Q}) \subset \mathrm{Pn}_{m+1}(\mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C},\mathfrak{Q})$ to be the full subspace spanned by the locally $(m+1)$-bounded Poincaré objects (X,ξ) of $\mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C},\mathfrak{Q})$ satisfying the following:

   (a) The restriction of Y along $\mathcal{Q}_{\mathfrak{P}^\circ([n])}(\mathcal{C},\mathfrak{Q}) \subset \mathcal{Q}_{\mathfrak{P}([n])}(\mathcal{C},\mathfrak{Q})$ is locally m-bounded;

   (b) the diagram $g_{(X,\xi)} : \mathfrak{P}([n]) \xrightarrow{\psi_{(X,\xi)}} \mathrm{Surg}(\mathcal{C},\mathfrak{Q})_{(x,q)} \to \mathcal{C}_{/x} \to \mathcal{C}$ is strongly cocartesian, and takes values in $\Sigma^{-m-1}\mathcal{C}^{h\heartsuit} \subset \mathcal{C}$.

Since $\varnothing \in \mathfrak{P}([n])$ is sent to 0 by $g_{)X,\xi)}$, the vertices of the strongly cocartesian cube $g_{(X,\xi)}$ are given as direct sums of the $z_i := z_{\{i\}}$, for $0 \le i \le n$.

(iv) Varying n, we obtain a semisimplicial space $\mathrm{SC}^m(\mathcal{C},\mathfrak{Q})$, inducing a diagram of semisimplicial spaces

$$\begin{array}{ccccc} \mathrm{Pn}_m(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_m(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C},\mathfrak{Q}) \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \mathrm{SC}^m_\bullet(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C},\mathfrak{Q}) & \longrightarrow & \mathrm{Pn}_{m+1}(\mathcal{Q}_{\mathfrak{P}^\circ([\bullet])}(\mathcal{C},\mathfrak{Q}), \end{array}$$

which admits a dashed lift restricting along $\mathfrak{P}^\circ([n]) \subset \mathfrak{P}([n])$, and where the leftmost vertical map takes an m-bounded Poincaré object (x,q) to the associated constant cube. It thus suffices to show that the composite map $|\mathrm{SC}^m_\bullet(\mathcal{C},\mathfrak{Q})| \to |\mathrm{Pn}_{m+1}(\mathcal{Q}_{\mathfrak{P}([\bullet])}(\mathcal{C},\mathfrak{Q})| \simeq \mathrm{Pn}_{m+1}(\mathcal{C},\mathfrak{Q})$ induced levelwise by the map $\mathrm{SC}^m_\bullet(\mathcal{C},\mathfrak{Q}) \to \mathrm{Pn}_{m+1}(\mathcal{C},\mathfrak{Q})$ sending a cube of surgery data to its inital vertex is an equivalence.

(v) This is checked fibrewise, which reduces to showing that the realisation of the semisimplicial space $\mathrm{fib}_{(x,q)}(\mathrm{SC}^m_\bullet(\mathcal{C},\mathfrak{Q}) \to \mathrm{Pn}_{m+1}(\mathcal{C},\mathfrak{Q})) =: F^m_\bullet$ is contractible for each $(m+1)$-bounded Poincaré object of $(\mathcal{C},\mathfrak{Q})$. This in turn reduces by the argument of [HS25, Lem. A.1.13] to showing the following:

(i) F^m_0 is non-empty;

(ii) $F^m_1 \to F^m_0 \times F^m_0$ has non-empty fibres;

(iii) $F^m_\bullet$ is 1-coskeletal.

While (iii) is shown to follow ultimately from 2-excisivity of $\mathfrak{Q}$, the first two criteria follow from connectivity considerations owing to the weight structure on $\mathcal{C}$, key to which is the assumption that $\mathfrak{Q}$ takes connective values when restricted to $\mathcal{C}^{h\heartsuit}$. It is now time for the speculation.

Remark C.5.4.

(i) Suppose given an ordinary additive category with duality $(\mathcal{A}, \mathbb{D}_\mathcal{A})$ (say $(\mathrm{Proj}_R, [-, R])$). The duality $\mathbb{D}_\mathcal{A}$ derives to a duality on the stable envelope $K_b(\mathcal{A})$, with respect to which we may consider the symmetric and genuine symmetric Poincaré structure. The restriction of the connective cover $\tau_{\geq 0}\mathfrak{Q}^s|_\mathcal{A}$ of the symmetric forms functor along $\mathcal{A} \subset K_b(\mathcal{A})$ coincides with that of the genuine symmetric forms $\tau_{\geq 0}\mathfrak{Q}^{gs}|_\mathcal{A}$ and is additively quadratic, whilst in general, the map $\mathcal{L}^{gs}(\mathcal{A}) \to \mathcal{L}^s(\mathcal{A})$ is not an equivalence; see [Rea24] for a proof that $\mathcal{L}^{gs}(\mathbb{Z}/4) \not\simeq \mathcal{L}^s(\mathbb{Z}/4)$.

(ii) We suspect that the analogue of the connectivity condition on $\mathfrak{Q}$ for stable ∞-categories equipped with bounded heart structures is that $\mathfrak{Q}$ has connective 1-excisive approximation when this is viewed as an object of $\mathrm{Fun}^{ex}(\mathcal{C}^{op}, \mathcal{S}p) \simeq \mathcal{S}h^{\mathcal{S}p}_\Sigma(\mathcal{C}^{h\heartsuit})$ with respect to the Postnikov t-structure. Indeed, when $\mathcal{C}$ is weighted (i.e. $\mathcal{C}^{h\heartsuit}$ is split exact), connectivity in $\mathcal{S}h^{\mathcal{S}p}_\Sigma(\mathcal{C}^{h\heartsuit}) = \mathcal{P}^{\mathcal{S}p}_\Sigma(\mathcal{C}^{h\heartsuit})$ is a pointwise criterion, so the fibre sequence $B_\mathfrak{Q}(x,x)_{hC_2} \to \mathfrak{Q}(x) \to P_1\mathfrak{Q}(x)$ for $x \in \mathcal{C}^{h\heartsuit}$ and connectivity of the restricted mapping spectra of $\mathcal{C}^{h\heartsuit}$ imply that $\mathfrak{Q}$ takes connective values on $\mathcal{C}^{h\heartsuit}$ if and only if $P_1\mathfrak{Q}$ is connective on $\mathcal{C}^{h\heartsuit}$.

(iii) Under this assumption, we note the following: the space F^m_n above has vertices tuples

$$(f_i : z_i \to x)_{0\leq i\leq n}, \eta),$$

where $z_i \in \Sigma^{-m-1}\mathcal{C}^{h\heartsuit}$, $\mathrm{cofib}(f_i) \in \mathcal{C}_{\geq -m}$, and η is a nullhomotopy of the image of the form q in $\Omega^\infty\mathfrak{Q}(\bigoplus_i z_i)$. To show F^m_0 is non-empty we use a heart decomposition $z \xrightarrow{f} x \to y$ with $z \in \mathcal{C}_{\leq -m-1}$, $y \in \mathcal{C}_{\geq -m}$, so that since $x \in \mathcal{C}_{[-m-1,m+1]}$, $z \in \Sigma^{-m-1}\mathcal{C}^{h\heartsuit}$. Recall from Remark C.1.5 that the presheaves $\pi_n \hom_\mathcal{C}(-, w)$ for $w \in \mathcal{C}^{h\heartsuit}$ are weakly effaceable for $n < 0$ when restricted to $\mathcal{C}^{h\heartsuit}$. We have a diagram of abelian groups

$$\begin{array}{ccccc}
\pi_0 B_\mathfrak{Q}(x,x)_{hC_2} & \longrightarrow & \pi_0\mathfrak{Q}(x) & \longrightarrow & \pi_0\Lambda_\mathfrak{Q}(x) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0 B_\mathfrak{Q}(z,z)_{hC_2} & \longrightarrow & \pi_0\mathfrak{Q}(z) & \longrightarrow & \pi_0\Lambda_\mathfrak{Q}(z)
\end{array}$$

with exact rows, with $\pi_0\Lambda_\mathfrak{Q}(z)$ weakly effaceable by assumption since $z \simeq \Sigma^{m+1}z_0$ for some $z_0 \in \mathcal{C}^{h\heartsuit}$ and $m \geq 0$. Since also the presheaves $B_\mathfrak{Q}(-,-) \simeq \hom_{\mathcal{C}^{h\heartsuit}}(-, \mathbb{D}_\mathfrak{Q}(-))$ are weakly effaceable in negative homotopy degrees, the presheaf $x \mapsto B_\mathfrak{Q}(x,x)_{hC_2}$ is weakly effaceable in negative degrees by Lemma C.1.6. A diagram chase then gives that $\pi_0\mathfrak{Q}(z)$ is weakly effaceable, so for some map $p : z' \twoheadrightarrow z$ whose image under Σ^{m+1} is an egression we have $p^*f^*[q] = 0$. The congression of cofibres

$$\mathrm{cofib}(z' \twoheadrightarrow z) \simeq \Sigma\,\mathrm{fib}(z' \twoheadrightarrow z) \rightarrowtail \mathrm{cofib}(z' \to x) \twoheadrightarrow \mathrm{cofib}(z \to x)$$

and $(-m)$-connectivity of y implies that the middle term is $(-m)$-connective, since $\mathrm{fib}(z' \twoheadrightarrow z) \in \Sigma^{-m-1}\mathcal{C}^{h\heartsuit}$.

(iv) To check the remaining criterion we need to exhibit, given $(f_i : z_i \to x, \eta_i : f_i^* q \simeq 0)_{i=0,1}$, a null-homotopy of the image of q in $\Omega^\infty Ϙ(z_0 \oplus z_1)$, extending the η_i. This doesn't quite follow from our assumptions: using that $Ϙ(z_0 \oplus z_1) \simeq Ϙ(z_0) \oplus Ϙ(z_1) \oplus B_Ϙ(z_0, z_1)$, the image of q upon restriction to $B_Ϙ(z', z_1)$ for some shifted egression $z' \twoheadrightarrow z_0$ vanishes, but without the uniform effaceability imposed by the connectivity axiom for weight structures, it is not clear how to proceed.

It may be the case that working with a variant of the surgery complex $\mathrm{SC}^m_\bullet(\mathcal{C}, Ϙ)$ may push the argument over the line. Note that vanishing of the image of $[q]$ in $B_Ϙ(z', z_1)$ is equivalent to vanishing of the composite

$$z' \overset{p}{\twoheadrightarrow} z_0 \xrightarrow{f_0} x \xrightarrow{q_\#} \mathbb{D}_Ϙ(x) \xrightarrow{\mathbb{D}_Ϙ(f_1)} \mathbb{D}_Ϙ(z_1),$$

and it is tempting to ask for conditions on an exact ∞-category $\mathcal{E}$ such that egressions are epimorphisms in its stable hull, but this is the case only when the exact structure on $\mathcal{E}$ is split.

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