READING GROUP ON INTERNAL HIGHER CATEGORIES: INTRODUCTION

0.1. **Motivation.** Consider for a profinite group G the category $\mathbf{O}_G$ of transitive continuous G-sets and G-equivariant maps. Elmendorf's theorem furnishes an equivalence $\mathcal{S}_G \simeq \mathrm{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathcal{S})$ between the ∞-category of G-spaces (the underlying ∞-category of the model category of G-simplicial sets in which weak equivalences are detected on H-fixed points for each closed $H \leq G$), and the category of presheaves of spaces on $\mathbf{O}_G$, or equivalently cartesian fibrations over $\mathbf{O}_G$ (with morphisms functors over $\mathbf{O}_G$ preserving cartesian edges). This refines further to a presheaf of categories

$$\underline{\mathcal{S}_G} : \mathbf{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad G/H \mapsto \mathrm{Fun}(((\mathbf{O}_G)_{/(G/H)})^{\mathrm{op}}, \mathcal{S}) = \mathrm{Fun}(\mathbf{O}_H^{\mathrm{op}}, \mathcal{S}),$$

where the decorations denote slice categories, and we use the equivalence $(\mathbf{O}_G)_{/(G/H)} \simeq \mathbf{O}_H$, the G-category of G-spaces. Now $\mathcal{S}_G$ is the free cocompletion of $\mathbf{O}_G$, and likewise one may show that $\underline{\mathcal{S}_G}$ is the free G-cocompletion of the (G-)point, i.e. the terminal G-category $* : \mathbf{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Similarly, we may consider the G-category of G-spectra. Links to global and parametrised homotopy theory:

- (i) Write Glo for the Duskin nerve of the $(2,1)$-category of finite groups, group homomorphisms, and conjugations[1]. One may define a Glo-category $\underline{\mathcal{S}_{\mathsf{Glo}}}$ of Glo-spaces, coinciding with the model categorical notion.
- (ii) One may consider spaces 'parametrised' by a base space X, i.e. functors $X \to \mathcal{S}$ satisfying suitable conditions.

One may consider for some base ∞-category T functors $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$; such T-categories assemble into a T-category $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $t \mapsto \mathrm{Fun}(T_{/t}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. Now since Cat_∞ is complete, we have a chain of natural equivalences

$$\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Fun}(T, \mathrm{Cat}_\infty^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^L(\mathcal{P}(T), \mathrm{Cat}_\infty^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^R(\mathcal{P}(T)^{\mathrm{op}}, \mathrm{Cat}_\infty) = \mathcal{S}\mathrm{hv}(\mathcal{P}(T); \mathrm{Cat}_\infty).$$

Note that $\mathcal{P}(T)$ is an ∞-topos, and in particular enjoys the property that the straightening of the codomain cartesian fibration $\mathcal{P}(T)^{\Delta^1} \to \mathcal{P}(T)$ is limit preserving, restricting to the presheaf $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $t \mapsto \mathrm{Fun}(T_{/t}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(T_{/t}) \simeq \mathcal{P}(T)_{/t}$. All of the fundamental theory of T-categories can be developed in this manner for an ∞-topos $\mathcal{B}$, in which setting one may avail oneself of the internal logic of the ∞-topos $\mathcal{B}$.

0.2. **∞-topoi and étale geometric morphisms.** For U an object of an ∞-topos $\mathcal{X}$, the slice category $\mathcal{X}_{/U}$ is again an ∞-topos and the forgetful functor $\pi_! : \mathcal{X}_{/U} \to \mathcal{X}$ admits a right adjoint $\pi^* : \mathcal{X} \to \mathcal{X}_{/U}$, $V \mapsto (V \times U \to U)$, which is colimit preserving and hence admits a further right adjoint $\pi_* : \mathcal{X}_{/U} \to \mathcal{X}$. A geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is said to be étale if it arises via the above construction in the sense that it admits a factorisation

$$\mathcal{X} \xrightarrow{f} \mathcal{Y}_{/U} \xrightarrow{\pi_*} \mathcal{Y}$$

for U an object of $\mathcal{Y}$ and f a categorical equivalence. We call an algebraic morphism $f^* : \mathcal{X} \to \mathcal{Y}$ étale if the corresponding geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$ is étale. By [HTT, Th. 6.5.3.11], this is equivalent to the following:

- (i) f^* admits a conservative left adjoint $f_!$;
- (ii) for each $X \to Y$ in $\mathcal{X}$, $Z \in \mathcal{Y}$, and map $f_!(Z) \to Y$, the induced diagram

$$\begin{array}{ccc} f_!(f^*(X) \times_{f^*(Y)} Z) & \longrightarrow & f_!(Z) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is cartesian, i.e. the canonical map $f_!(f^*(X) \times_{f^*(Y)} Z) \to X \times_Y f_!(Z)$ is an equivalence.

[1] I.e., map $\phi \Rightarrow \psi$ of maps $H \to G$ is an element $g \in G$ with $\phi = g\psi g^{-1}$.

0.3. **(Internal) orthogonality and factorisation systems.**

Definition 0.1. (i) Given maps $f : x \to y$ and $g : z \to w$ in an ∞-category $\mathcal{C}$, f is left orthogonal to g, denoted $f \perp g$, if the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(y,z) & \xrightarrow{f^*} & \mathrm{Map}_{\mathcal{C}}(x,z) \\ \downarrow g_* & & \downarrow g_* \\ \mathrm{Map}_{\mathcal{C}}(y,w) & \xrightarrow{f^*} & \mathrm{Map}_{\mathcal{C}}(x,w) \end{array}$$

of spaces is cartesian; this implies in particular that given the data of a solid commutative diagram

$$\begin{array}{ccc} x & \longrightarrow & z \\ \downarrow f & \nearrow & \downarrow g \\ y & \longrightarrow & w \end{array}$$

there exists an essentially unique dashed filler. Given a set $S \subset \mathcal{C}_1$ of morphisms in $\mathcal{C}$, write ${}^{\perp}S$ resp. $S^{\perp}$ for the class of maps in $\mathcal{C}$ which are left resp. right orthogonal to each map in S.

(ii) A factorisation system on $\mathcal{C}$ is the data of two classes $(\mathcal{L}, \mathcal{R})$ of morphisms in $\mathcal{C}$, such that
  (i) each map f in $\mathcal{C}$ admits a factorisation $f \simeq r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$;
  (ii) $\mathcal{L}^{\perp} = \mathcal{R}$ and ${}^{\perp}\mathcal{R} = \mathcal{L}$.

(iii) If $\mathcal{C}$ is cartesian closed with internal mapping object $[-,-] \in \mathcal{C}$, maps $f : x \to y$ is internally left orthogonal to $g : z \to w$, denoted $f \underline{\perp} g$, if the square

$$\begin{array}{ccc} [y,z] & \xrightarrow{f^*} & [x,z] \\ \downarrow g_* & & \downarrow g_* \\ [y,w] & \xrightarrow{f^*} & [x,w] \end{array}$$

in $\mathcal{C}$ is cartesian.

Proposition 0.2. *Suppose $\mathcal{C}$ is presentable, and $S \subset \mathcal{C}$ is a small set of maps.*

(i) [HTT, Prop. 5.5.5.7] There exists a factorisation system $(\mathcal{L}, \mathcal{R}$ on $\mathcal{C}$ with $\mathcal{R} = S^{\perp}$ (and $\mathcal{L} = {}^{\perp}\mathcal{R}$).

(ii) [ABFJ20, Prop. 3.2.9] Suppose $\mathcal{C}$ is moreover cartesian closed. Then there is a factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ with $\mathcal{R} = S^{\underline{\perp}}$ and $\mathcal{L} = {}^{\underline{\perp}}\mathcal{R} = {}^{\perp}\mathcal{R}$.

0.4. **$\mathcal{B}$-categories and -groupoids.** For $\mathcal{B}$ an ∞-topos, recall that there is an adjunction $(\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}) : \mathcal{S} \leftrightarrows \mathcal{B}$, with const the colimit-preserving extension of the functor $1_{\mathcal{B}} : * \to \mathcal{B}$ classifying the terminal object, and $\Gamma_{\mathcal{B}} = \mathrm{Map}_{\mathcal{B}}(1_{\mathcal{B}}, -)$. Universality of colimits in $\mathcal{B}$ implies that $\mathcal{B}$ is cartesian closed, i.e. for each $X \in \mathcal{B}$, the functor $X \times - : \mathcal{B} \to \mathcal{B}$ preserves colimits, and accordingly by the adjoint functor theorem admits a right adjoint $[X, -] : \mathcal{B} \to \mathcal{B}$. Now the category of simplicial objects $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{B}) \coloneqq \mathcal{B}_{\mathbb{\Delta}}$ in $\mathcal{B}$ is an ∞-topos, and we observe that the adjunction $(\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}})$ yields upon postcomposition an adjunction $(\mathrm{const}_{\mathcal{B},*} \dashv \Gamma_{\mathcal{B},*}) : \mathcal{S}_{\mathbb{\Delta}} \leftrightarrows \mathcal{B}_{\mathbb{\Delta}}$; we will generally omit mention of the functors $\mathrm{const}_{\mathcal{B}}$, $\mathrm{const}_{\mathcal{B},*}$.

Definition 0.3. A simplicial object $X_\bullet \in \mathcal{B}_{\mathbb{\Delta}}$ is a $\mathcal{B}$-category if it is internally orthogonal to $I^2 \hookrightarrow \Delta^2$ (Segal condition), and to $E^1 \to \Delta^0$ (univalence), where E^1 is the walking equivalence $(\Delta^0 \amalg \Delta^0) \coprod_{(\Delta^1 \amalg \Delta^1)} \Delta^3$. $X_\bullet$ is a $\mathcal{B}$-groupoid if it is internally local with respect to $\Delta^1 \to \Delta^0$, and we write $\mathrm{Grpd}(\mathcal{B}) \subset \mathrm{Cat}(\mathcal{B}) \subset \mathcal{B}_{\mathbb{\Delta}}$ for the corresponding full subcategories.

Equivalently, $X_\bullet$ is a $\mathcal{B}$-category if the Segal maps $X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ are equivalences for each $n \geq 2$, and the diagram

$$\begin{array}{ccc} C_0 & \longrightarrow & C_3 \\ \downarrow \Delta & & \downarrow d_{0,2}, d_{1,3} \\ C_0 \times C_0 & \longrightarrow & C_1 \times C_1 \end{array}$$

is cartesian in $\mathcal{B}$, by [Mar21, Prop. 3.2.7]. Here, $d_{i,j}$ is induced by the unique injective map $[1] \to [3]$ in $\mathbb{\Delta}$ omitting i and j. By [Mar21, Cor. 3.2.12], a simplicial object C is a $\mathcal{B}$-groupoid if and only if it is in the essential image of the diagonal embedding $\iota : \mathcal{B} \hookrightarrow \mathcal{B}_{\mathbb{\Delta}}$. The inclusion $\mathrm{Cat}(\mathcal{B}) \subset \mathcal{B}_{\mathbb{\Delta}}$ preserves limits and accordingly admits a left adjoint which preserves finite products; it follows formally that $\mathrm{Cat}(\mathcal{B})$ is an exponential ideal in $\mathcal{B}_{\mathbb{\Delta}}$ and is accordingly cartesian closed; write $\underline{\mathrm{Fun}}_{\mathcal{B}}(-,-)$ for the internal mapping object.

Given a morphism of topoi $(f^*, f_*) : \mathcal{A} \leftrightarrows \mathcal{B}$, we obtain by postcomposition an adjunction $(f^*, f_*) : \mathcal{A}_{\mathbb{\Delta}} \leftrightarrows \mathcal{B}_{\mathbb{\Delta}}$, which since f^* is left exact restricts to the full subcategories $\mathrm{Cat}(\mathcal{A})$ and $\mathrm{Cat}(\mathcal{B})$. In particular, for each ∞-topos $\mathcal{B}$ we have an adjunction

$$\mathrm{Cat}_\infty \underset{\Gamma_{\mathcal{B}}}{\overset{\mathrm{const}_{\mathcal{B}}}{\rightleftarrows}} \mathrm{Cat}(\mathcal{B}).$$

Definition 0.4. Define bifunctors

$$\begin{aligned} \mathrm{Fun}_{\mathcal{B}} &\coloneqq \Gamma_{\mathcal{B}} \circ \underline{\mathrm{Fun}}_{\mathcal{B}} : \mathrm{Cat}_\infty \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}), \\ (-) \otimes (-) &\coloneqq \mathrm{const}_{\mathcal{B}}(-) \times (-) : \mathrm{Cat}_\infty \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}), \\ (-)^{(-)} &\coloneqq \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathrm{const}_{\mathcal{B}}, -) : \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}), \end{aligned}$$

satisfying

$$\mathrm{Map}_{\mathrm{Cat}(\mathcal{B})}(\mathcal{C} \otimes \mathsf{D}, \mathsf{E}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})) \simeq \mathrm{Map}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{D}, \mathsf{E}^{\mathcal{C}}).$$

for $\mathcal{C} \in \mathrm{Cat}_\infty$ and $\mathsf{D}, \mathsf{E} \in \mathrm{Cat}(\mathcal{B})$

Since $\mathcal{B}$ is an ∞-topos, the Yoneda embedding factors through an equivalence $\mathcal{B} \simeq \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathcal{S})$. The composite

$$\mathcal{B}_{\mathbb{\Delta}} \subset \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{B}) \simeq \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathcal{S}_{\mathbb{\Delta}}) =: \mathcal{S}\mathrm{hv}(\mathcal{B}; \mathcal{S}_{\mathbb{\Delta}})$$

identifies $\mathcal{B}_{\mathbb{\Delta}}$ with the category of sheaves of ∞-categories on $\mathcal{B}$, via the identification $N_\bullet \coloneqq \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, -) : \mathrm{Cat}_\infty \simeq \mathrm{CSS}(\mathcal{S}) \subset \mathcal{S}_{\mathbb{\Delta}}$ ($N_\bullet$ is the Rezk nerve). This equivalence informally sends a $\mathcal{B}$-category $\mathsf{C} \in \mathcal{B}_{\mathbb{\Delta}}$ to the sheaf of complete Segal spaces/categories $\mathrm{Map}_{\mathcal{B}}(-, \mathsf{C}_\bullet)$. Under the equivalence $(\mathsf{C}^{\Delta^n})_0 \simeq \mathsf{C}_n$ of [MW24, Rem. 2.6.9], this corresponds under the equivalence $\mathcal{S}\mathrm{hv}(\mathcal{B}; \mathrm{Cat}_\infty) \simeq \mathcal{S}\mathrm{hv}(\mathcal{B}; \mathrm{CSS}(\mathcal{S}))$ induced by the Rezk nerve, this corresponds to the $\mathrm{Fun}_{\mathcal{B}}(\iota(-), \mathsf{C}) : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ [Mar21, §3.5], via the equivalences

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, \mathrm{Fun}_{\mathcal{B}}(\iota(-), -)) \simeq \mathrm{Map}_{\mathrm{Cat}(\mathcal{B})}(\iota(-), (-)^{\Delta^\bullet}) \simeq \mathrm{Map}_{\mathcal{B}}(-, (-)_\bullet).$$

0.5. **Contexts and mapping groupoids.** For A an object of $\mathcal{B}$ and C a $\mathcal{B}$-category, write $\mathsf{C}(A) \coloneqq \mathrm{Fun}_{\mathcal{B}}(\iota(A), \mathsf{C})$ for the local sections over A. For an integer $n \geq 1$, define an n-morphism in context A as any of the equivalent data:

(i) a map $\Delta^n \to \mathsf{C}(A)$ in Cat_∞;
(ii) a map $\iota(A) \to \mathsf{C}^{\Delta^n}$ in $\mathrm{Cat}(\mathcal{B})$;
(iii) a map $\Delta^n \otimes \iota(A) \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$;
(iv) a map $A \to \mathsf{C}_n$ in $\mathcal{B}$.

Given two 0-morphisms (objects) x, y in context A, define the mapping $\mathcal{B}_{/A}$-groupoid as the left vertical leg in the cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathsf{C}}(x,y) & \longrightarrow & \mathsf{C}_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_1, d_0)} \\ A & \xrightarrow{(x,y)} & \mathsf{C}_0 \times \mathsf{C}_0. \end{array}$$

A section $A \to \mathrm{Map}_{\mathsf{C}}(x,y)$ is then the data of a 1-morphism (map) f in context A, classified by $A \to \mathsf{C}_1$, from x to y. An n-morphism in global context is classified by a map $1_{\mathcal{B}} \to \mathsf{C}_n$ from the terminal object of $\mathcal{B}$; note that the geometric morphism $((\pi_A)_!, \pi_A^*)$ allows us to base change to a global context in the sense that the data of a map $A \to \mathsf{C}^{\Delta^n}$ in $\mathrm{Cat}(\mathcal{B})$ is equivalent to that of a map $1_{\mathcal{B}_{/A}} \to \pi_A^*(\mathsf{C})^{\Delta^n} \simeq \pi_A^*(\mathsf{C}^{\Delta^n})$ in $\mathrm{Cat}(\mathcal{B}_{/A})$.

References

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